Intrinsic Linking of Complete Partite Graphs

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Abstract

I will discuss the conjecture mentioned in Adams' The Knot Book that removing any vertex from an intrinsically knotted graph results in an intrinsically linked graph. Although Foisy recently showed that the conjecture doesn't hold in full generality, it remains open for specific classes of graphs. I will discuss the case of complete partite graphs and show that the conjecture does in fact hold for this class of graphs.

This was joint work with Thomas Mattman that began during the summer of 2003 and continued through the 2003/2004 school year as an honors project. Throughout working on this project, I presented my research to a variety of conferences and competitions. The funding to allow me to travel to these events was largely provided by the CSU, Chico Department of Mathematics and Statistics. In addition, I received a funding from the CSU, Chico Research foundation and the College of Natural Sciences for the Joint Meetings in January 2004. I also received a travel grant from the MAA for Mathfest 2003 and the conference at The Ohio State University in summer of 2003 was fully funded by Ohio State.

Acknowledgments

First of all, I would like to thank Dr. Thomas Mattman. It was Dr. Mattman who first introduced me to research during the summer of 2003 and who suggested I extend that research to an honors project for the following academic year. Without him, I would never have gotten started on this project, and his amazing insight and patience was instrumental in the completion of the project. In addition to this, Dr. Mattman has done an excellent job of finding conferences where I could present my work. Although it was very nerve racking at first, I now have a good deal of experience presenting and I actually look forward to it. I also extend my thanks to Dr. Margaret Owens and Dr. Robin Soloway for reading my thesis. It is only the most recent of the many things these professors have done to help me throughout my undergraduate career.

I would also like to thank all of my professors at CSU, Chico for helping me develop my mathematical skills. I would list the most influential professors; however, this would lead me to list nearly every professor in the department. It is amazing that there are so many amazing teachers all in one place, I thank you all.

Chapter 1

Introduction

In this research project, I explored an unsolved problem posed in The Knot Book by Colin Adams [A94]. One of the reasons I chose this problem is that it combines ideas from both knot theory and graph theory, and combining ideas from different disciplines often produces very interesting results. Chapter 1 will be an introduction to graph theory and knot theory, chapter 2 will be the statement of the problem, and chapter 3 will be the actual proof of that problem. I am assuming the reader has a strong mathematical background, for example, has completed the junior year of a math degree. A course in topology would be helpful for some of the more technical aspects of the paper; however it is not required to understand the main points of my thesis.

1.1 Introduction to Graph Theory

Material in this section could be found in most introductory texts on graph theory, for example [KPV03].

When you think of a graph, one of the objects in figure 1.1 might come to mind.

In graph theory however, a graph is not one of these familiar objects.

Definition 1 *A graph is a finite set V of vertices and a set $E \subset V \times V$ of edges. We will say that vertices v_1 and v_2 are connected by edge (v_1,v_2), and that edge (v_1,v_2) is incident to both v_1 and v_2.*

An example of a graph is shown in figure 1.2.

Definition 2 *When we refer to deleting an edge e from graph G with edges E and vertices V, we*

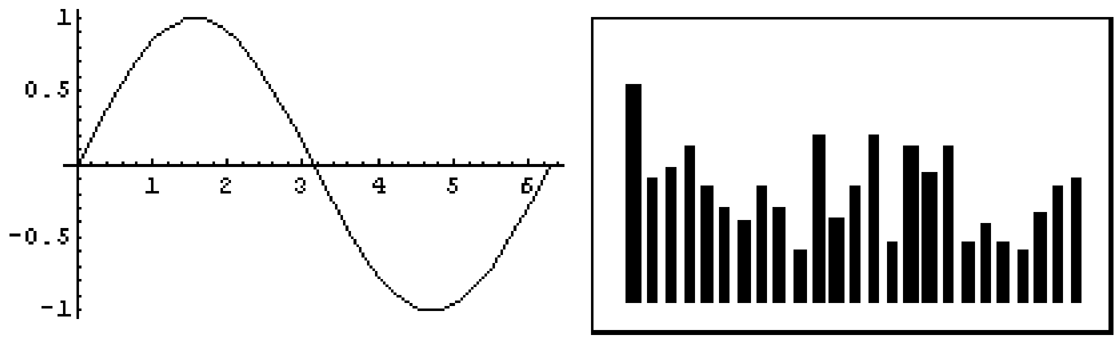

Figure 1.1: Common Graphs

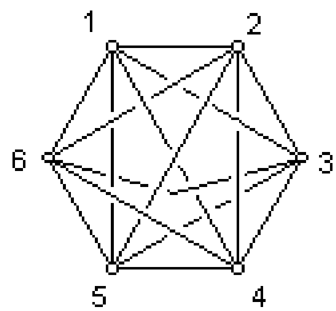

Figure 1.2: The K_6 Graph

mean to consider a subgraph G' in which the set of vertices of G' is V, and the set of edges of G' is $E \backslash \{e\}$.

Definition 3 *When we refer to deleting a vertex v from graph G with edges E and vertices V, we mean to consider a subgraph G' in which the set of vertices of G' is $V \backslash \{v\}$, and the set of edges of G' is $E \backslash \{e | e \text{ is incident to } v\}$.*

Oddly enough, the placement of the vertices and the length and shape of the edges is not stated in the description of the graph. For example, the graph pictured in figure 1.2 is called K_6, and is an example of a complete graph.

Definition 4 *In a complete graph, any pair of vertices is connected by an edge. A complete graph with n vertices is denoted K_n.*

Any other graph that has six vertices and an edge between every pair of vertices would also be called K_6. A few more examples of the K_6 graph are shown in figure 1.3.

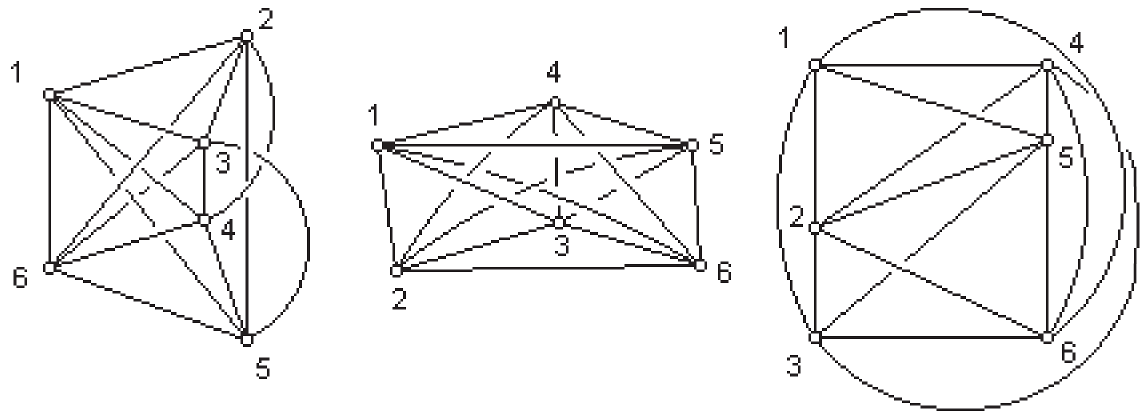

Figure 1.3: More examples of K_6

At first sight, it might seem strange that several different looking things represent the same graph. Although the graphs pictured in figures 1.2 and 1.3 are all the same graph, they are different embeddings of that graph.

Definition 5 *An embedding of a graph is a way of placing the graph in $\mathbb{R}^3$. Vertices are represented by distinct points in $\mathbb{R}^3$. Edge (v_1, v_2) is represented by an arc between v_1 and v_2. Edges with a common endpoint intersect only at that common endpoint, edges without common endpoints do not intersect anywhere.*

Note: The notion of an arc used in this definition is a fairly intuitive concept; however, it is defined formally below in definition 17.

The pictures in figures 1.2 and 1.3 represent the same graph because they have the same number of vertices connected by edges in the same way, however they are different embeddings of the graph because they are placed differently in space.

Definition 6 *The graphs in figures 1.2 and 1.3 are actually two dimensional representations of three dimensional objects that we call projections. For an embedding of a graph, we do not allow edges to intersect, so when two lines cross over another, we illustrate this in the projection by having the line on top solid, and the line underneath broken. In a projection, no more than 2 lines can cross at a single point.*

Note: The term refers to projecting the graph onto a plane. Given a plane P and graph G, an arbitrarily small perturbation of P or G ensures projection is a homeomorphism except at a finite number of double points where 2 edges cross transversally (See section 3.E of Rolfsen [R90]). Projection for knots and links, which are defined in the next section, is defined in a similar way.

Definition 7 *We will call a graph G* <u>*planar*</u> *if there exists an embedding of G that can be drawn completely in the plane. Conversely, a graph for which no embeddings can be drawn in a plane is* <u>*non-planar*</u>*.*

Notice that if G is planar, there will exist a projection of G with no crossings.

In this paper, I will be working with a class of graphs called complete partite graphs.

Definition 8 *In a* <u>*complete partite graph*</u>*, the vertices are partitioned into disjoint subsets, called parts. Any two vertices in different parts are connected by an edge, while vertices in the same part are not connected to one another.*

A complete partite graph with n parts is denoted $K_{a_1,a_2,...,a_n}$ where the subscripts give the number of vertices in each part. Several examples of complete partite graphs are shown in figure 1.4; vertices with the same label belong to the same part. We will consider $K_{a_1,a_2,...,a_n}$ the same as $K_{b_1,b_2,...,b_n}$ if $\{a_1,a_2,...,a_n\}$ is a permutation of $\{b_1,b_2,...,b_n\}$. Generally we will write the parts in descending order.

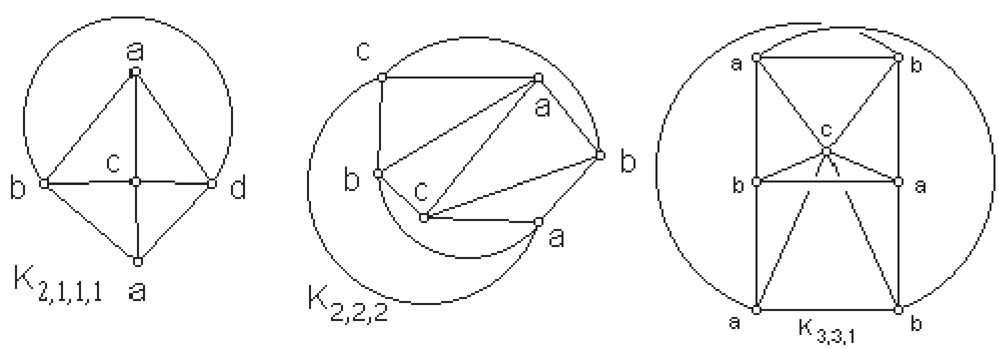

Figure 1.4: Examples of Complete Partite Graphs

Definition 9 *Graph G' is obtained from graph G by* <u>*splitting a vertex*</u> *if it is obtained as follows. To split a vertex v, one must first delete vertex v. Then add vertices v_1 and v_2, and connect them by an edge. Then each vertex that was connected to v by an edge, must be connected to exactly one of v_1 or v_2 by an edge. An example is shown in figure 1.5.*

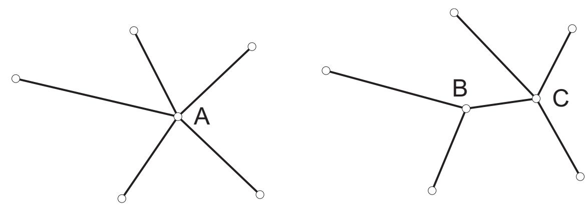

Figure 1.5: Vertex A Split to Vertices B and C

Notice that in most cases, there are multiple ways to split any vertex v. In fact, if there are n vertices connected to vertex v, there are 2^n ways to split vertex v. To see this, notice that each vertex connected to vertex v in G can be connected to either v_1 or v_2 in G′, so there are 2 possibilities for each of these connections in graph G′.

Definition 10 *An* *expansion* *of a graph G is a graph constructed from G by adding any number of vertices and edges and splitting any number of vertices. G is not an expansion of G.*

Definition 11 *An* *edge contraction* *is an operation on an edge e of a graph resulting in a graph with one less vertex and at least 1 less edge. To contract edge e, delete it and the two vertices v_1 and v_2 it connects. Then add a new vertex v and add an edge between v and each vertex that was connected to either v_1 or v_2. An example of an edge contraction is shown in figure 1.6.*

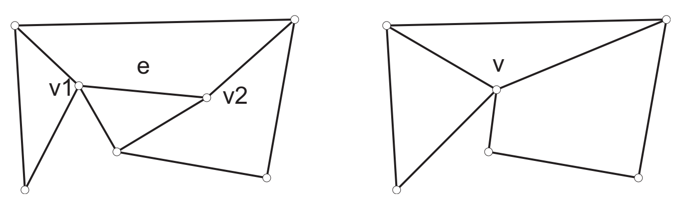

Figure 1.6: Contracting Edge e

Definition 12 *A* *minor* *of a graph G is a graph constructed by deleting any number of edges and vertices of G, and contracting any number of edges of G. G is not a minor of itself.*

Notice that minor is the opposite of expansion. So, G′ is a minor of G iff G is an expansion of G′.

Definition 13 *A graph G is* *minor minimal* *with respect to a property if G exhibits that property, yet no minor of G exhibits the property.*

I like to think of a graph that is minor minimal with respect to a property as a graph that barely possesses that property.

A simple example of this is that $K_{3,3}$ is minor minimal with respect to non-planarity. $K_{3,3}$ is known to be non-planar (theorem 5 below); however, the removal of an edge, vertex, or the contraction of an edge all result in a planar graph as pictured in figure 1.7.

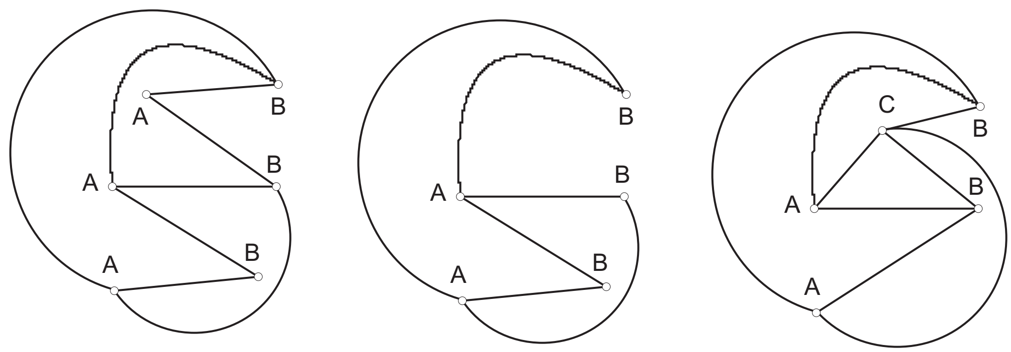

Figure 1.7: Planar minors of $K_{3,3}$

1.2 Introduction to Knot Theory

We begin with an informal overview of knot theory; a formal look will follow. Informally, a knot is simply a closed curve in space. To imagine a knot, visualize tying a piece of string and then connecting the ends. Several mathematical knots are pictured in figure 1.8, scanned from [A94].

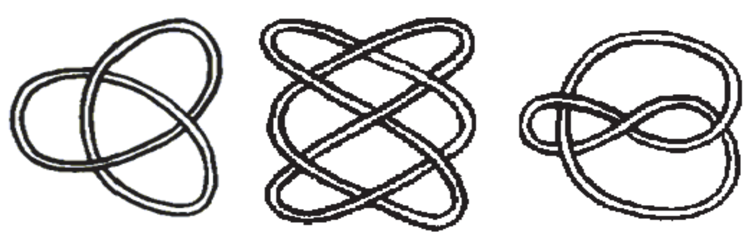

Figure 1.8: Examples of Knots

In the real world, it is natural to use an extension cord to simulate a knot because you can easily connect the ends by plugging it into itself. Once the ends are connected, we have a knot. As long as you don't unplug it or cut it anywhere you can move the cord around and we'll say it is still the same knot. For example, there are two common ways to draw a trefoil knot. A progression from one to the other is drawn in figure 1.9.

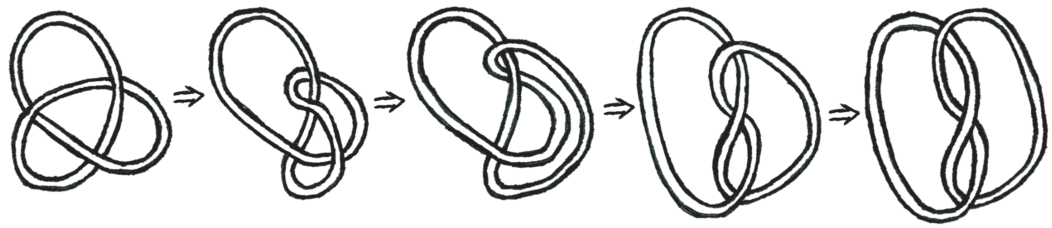

Figure 1.9: Trefoil Progression

Once the idea of a knot is understood, a link is fairly simple. A link differs from a knot in that a link is two or more ropes tangled together, while a knot is only one rope. A link is like a knot in that the ropes can be moved without changing the link. Several links are pictured in figure 1.10, scanned from [A94]. Recall definition 6; projections of knots and links are similar in that a knot or

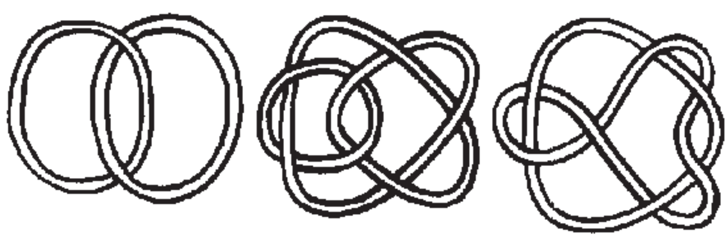

Figure 1.10: Examples of Links

link can not have any points of intersection. So when we project onto a plane, if 2 strands of the knot or link cross, we draw one solid to illustrate that it is above the other, and the other will be broken, to illustrate that it is under the first.

So now we have an idea of what knots and links are, and we can use these ideas to figure things out about them. However, although these ideas are useful, they are insufficient to produce a rigorous mathematical theory. As usual in mathematics, while the informal ideas are very useful for getting a general idea of what is going on, we need a more formal definition. Sometimes strange things happen and the informal ideas do not give us a clear idea of what exactly is going on. In such instances, we must appeal to the formal definitions. In the following section, I will define knots more formally.

1.2.1 A Formal Introduction to Knots

Along with knots and links, my goal will be to define isotopy, but first I need to define several other things.

Definition 14 *A topological space is a set, X, with a collection of open sets, Ω, where*

$\emptyset$,X$\in \Omega$,

If $\Phi, \Theta \in \Omega$, then $\Phi \cap \Theta \in \Omega$, and

If $\Phi_i \in \Omega$, then $\bigcup_i \Phi_i \in \Omega$.

An important example is X $= \mathbb{R}^n$ and $\Omega = \{\bigcup_i (B(\overrightarrow{x_i}, r_i))\}$ where $B(\overrightarrow{x_i}, r_i) = \{\overrightarrow{x} : |\overrightarrow{x} - \overrightarrow{x_i}| < r_i\}$.

Definition 15 *A function f: $X, \Omega_X \to Y, \Omega_Y$ is continuous if $\forall \Phi \in \Omega_Y, f^{-1}(\Phi) \in \Omega_X$.*

Definition 16 *A function f: $X \to Y$ is a homeomorphism if f is a bijection and continuous and f^{-1} is also continuous.*

Definition 17 *An arc is a homeomorphism γ:[0,1] $\to I \subset X$. We often call the image in X the arc.*

Definition 18 *S^1 is $\mathbb{R} \cup \{\infty\}$. Open sets are unions of open set of $\mathbb{R}$ and sets of the form (a,∞) $\cup$ $\{\infty\}$ $\cup$ ($-\infty$,b) where a,b $\in \mathbb{R}$.*

Definition 19 *$K \subset \mathbb{R}^3$ is a knot if it is homeomorphic to S^1.*

Remark: We will often refer to a knot as a simple closed curve.

Definition 20 *The unknot is the circle embedded in $\mathbb{R}^3$ in the x,y plane centered at (0,0,0) with radius 1. We will often refer to the unknot as the trivial knot.*

Definition 21 *$L \subset \mathbb{R}^3$ is a link if it is homeomorphic to a finite disjoint collection of S^1's.*

Definition 22 *The unlink (of 2 components) is the link in which both components are circles in the x,y plane with radius $\frac{1}{2}$, one centered at (-1,0,0) and the other centered at (1,0,0). We will often refer to the unlink as the trivial link.*

Definition 23 *For an oriented knot or link, each component has a direction associated with it, to illustrate this, we draw arrows on the projection. An example is shown in figure 1.11.*

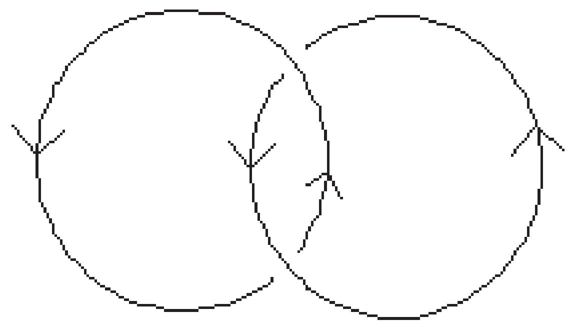

Figure 1.11: An Oriented Link

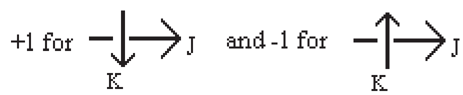

Figure 1.12: Counting for Linking Number

Definition 24 *Given a projection of a two component oriented link, call one component J and one K. Count each point where J crosses under K as shown in figure 1.12. The absolute value of the sum over all crossings of J under K is called the* <u>*linking number*</u>*, denoted lk(J,K).*

As an example, I will calculate the linking number of the link in figure 1.11. Call the component on the left J and the component on the right K. Notice that J crosses under K at one point, and according to figure 1.12, we add one for that crossing. So the total linking number is 1. Notice that if we change the orientation of J, the sign of the crossing changes, but the linking number is still 1 since we take the absolute value.

Definition 25 *Knots (or links) K, K' $\in \mathbb{R}^3$ are* <u>*equivalent*</u> *if there exists a homeomorphism h: $\mathbb{R}^3 \to \mathbb{R}^3$ such that h(K) = K'.*

Note that this is a homeomorphism of the whole space, and not just of the knot within that space.

Definition 26 *A* <u>*crossing change*</u> *is an operation on a crossing in a projection of a knot, link, or graph where we switch which strand crosses over and which crosses under. An example is shown in figure 1.13.*

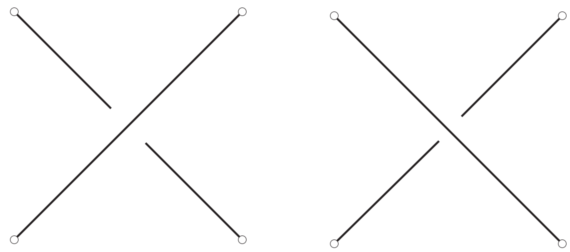

Figure 1.13: A Crossing Change

Note: a crossing change will generally produce a knot or link that is not equivalent to the original knot or link.

Definition 27 *A <u>tame</u> knot or link is one that is equivalent to an embedding of a polygon or a finite collection of polygons.*

We will only consider tame knots and links. We will also require arcs to be tame in the sense that they are a subset of some tame knot.

Definition 28 *For topological spaces* (X,Ω_X) *and* (Y,Ω_Y)*, the following is a description of the <u>product topology</u> on* $X \times Y$*.* $X \times Y$ *is a topological space with open sets* $\{\cup_i \Phi_i \times \Psi_i | \Phi_i \in \Omega_X, \Psi_i \in \Omega_Y\}$.

Definition 29 *A <u>homotopy</u> is a continuous function. h:* $X \times [0,1] \to X$.

Definition 30 *A homotopy h is called an <u>isotopy</u> if* $h(x,0) = x\ \forall x \in X$*, and for each t,* $h(*,t)=h|_{X\times\{t\}}$ *is a homeomorphism of X to itself.*

<u>Remark</u>: Informally, we often refer to the homeomorphism h($*$,1): X $\to$ X as an "isotopy". Intuitively, isotopy is the formal version of manipulating the rope which makes a knot or a link without cutting the rope. To isotope X (or a subset of X) means to apply the isotopy h($*$,1) to X (or its subset).

Notice that figure 1.9 is an example of an isotopy, and that very different looking things can actually be the same knot. This can make studying knot theory difficult because it is hard to know if two different looking objects are actually different knots or links, or if they are the same after some isotopy is performed. To help us overcome this difficulty, we have things called invariants.

Definition 31 *An <u>invariant</u> is an attribute, usually a number, associated with a knot or link that will not change through any isotopy of that knot or link.*

An example of an invariant is linking number, which is a link invariant.

We can follow one isotopy by another to get a third isotopy, i.e., given g_1, g_2, let t move through each of g_1 and g_2 "twice as fast" to produce a new isotopy h. Note that within h, g_2 needs to be composed with $g_1(*, 1)$, so that h(∗,1)=$g_2(*, 1) \circ g_1(*, 1)$. Also, we can compose an isotopy with a homeomorphism H of X to get an isotopy. g(x,t)=h(H(x),t) is again an isotopy; we'll write g=h∘H. Similarly, f(x,t)=H(h(x,t)) is an isotopy; we'll write f=H∘h.

1.3 Combining the Disciplines

In this project, I explore the idea of knots and links within graphs. A knot in a graph is simply some path of vertices and edges that forms a knot. Since a knot can't intersect itself, any vertex or edge can be used at most one time. Since a knot is a closed loop, the path through the graph must be closed. To find a link in a graph, the same process is used except we must now follow multiple disjoint paths through the graph. Notice that such knots or links in a graph depend on how the graph is embedded in $\mathbb{R}^3$. Different embeddings will result in non-equivalent knots or links.

Definition 32 *A path in a graph is a sequence of vertices connected by edges.*

Definition 33 *A simple path is one in which each vertex occurs at most once in the path, except possibly with the first and last vertex being equal. If the first and last vertex are equal, we will call the path closed.*

Definition 34 *A cycle in a graph is a simple closed path in the graph.*

Definition 35 *We will call an embedding of a graph knotted if it contains a cycle that forms a non-trivial knot.*

Definition 36 *We will call an embedding of a graph linked if it contains cycles that form a non-trivial link.*

Any graph that contains a closed loop will have knotted embeddings. However, some embeddings may exist which are not knotted. For example, two embeddings of K_3 are shown in figure 1.14, one is knotted, one is not.

Similarly, any graph containing two disjoint loops may be linked. However, there may be embeddings which are not linked. Two embeddings of $K_3 \cup K_3$ are shown in figure 1.15, one of which is linked, one of which is not.

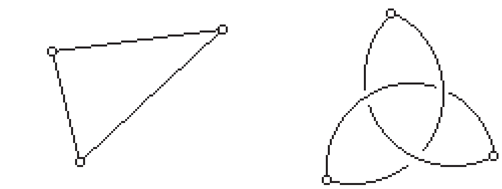

Figure 1.14: Two Embeddings of K_3

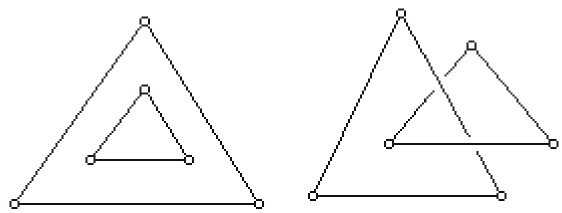

Figure 1.15: Two Embeddings of $K_3 \cup K_3$

Not all graphs can be drawn without a link; some graphs are linked regardless of the embedding.

Definition 37 *An intrinsically linked graph is one for which every embedding is linked.*

One such graph is K_6. I will now prove that K_6 is intrinsically linked, a result first proved by Conway and Gordon [CG83].

Theorem 1 (Conway and Gordon) *K_6 is intrinsically linked.*

Proof:

Given an embedding of K_6, define $\lambda \in \mathbb{Z}_2$ by

$$\lambda = \sum lk(C_1, C_2) mod 2$$

where we sum over all $\frac{1}{2}\binom{6}{3} = 10$ pairs of disjoint cycles contained in K_6.

Any embedding of K_6 can be obtained from any other through isotopy and crossing changes. While this is intuitively clear, it is difficult to prove. Recall that isotopy is the formal version of moving a rope of a link around without cutting it. Similarly, isotopy in graphs is a formal way to say we are moving the edges and vertices around without passing them through one another. Since isotopy does not change linking number, it will not change λ. I will now show that crossing changes also do not change λ.

Since the linking number does not count self crossings, a crossing of an edge with itself or with an adjacent edge will not change the linking number of any pair of loops in the graph, so it will not change λ. Every other crossing will be in exactly two pairs of disjoint cycles. To see this, consider any crossing of two separate edges. Notice this uses two edges and four vertices. It follows that there are two vertices left. Each edge of the crossing needs one more vertex to create a loop. Therefore, one edge uses one vertex and the other edge uses the other vertex. There are clearly two ways to do this, so there are two pairs of disjoint cycles which use the crossing. By changing the crossing, the linking number of each of the two pairs of loops will be changed by one, therefore, λ will be unchanged. (Recall, λ is defined mod 2.)

We have shown that λ is the same for all embeddings of K_6. Now we will calculate λ for a particular embedding of K_6. Consider the embedding in figure 1.16. Notice that there is only one

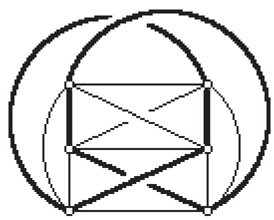

Figure 1.16: K_6 with the Link in Bold

non-trivial link contained in this embedding (highlighted in bold). The linking number of this link is 1, so λ is 1. This completes the proof. $\square$

Robertson, Seymour, and Thomas were able to find a list of all graphs that are minor minimal with respect to intrinsic linking, which resulted in the following theorem.

Theorem 2 *A graph is intrinsically linked iff it contains one of the seven Petersen graphs (shown in figure 1.17) as a minor [RST95].*

Note that the leftmost graph in the top row is $K_{3,3,1}$, the second from the left is K_6, and the graph in the middle of the second row is $K_{4,4} - \{e\}$. It is trivial to verify this for K_6, but the other two are a little harder. To see that the top left graph is $K_{3,3,1}$, notice that the lowest vertex is connected to everything, so it is the part with only 1 vertex. Of the remaining vertices, the top two vertices on the left and the bottom right vertex are the vertices from one of the parts with three vertices, and the remaining three are the other part. To see the bottom middle graph is $K_{4,4} - \{e\}$, pick one of the top 2 vertices, the four vertices connected to it constitute one part, the other four

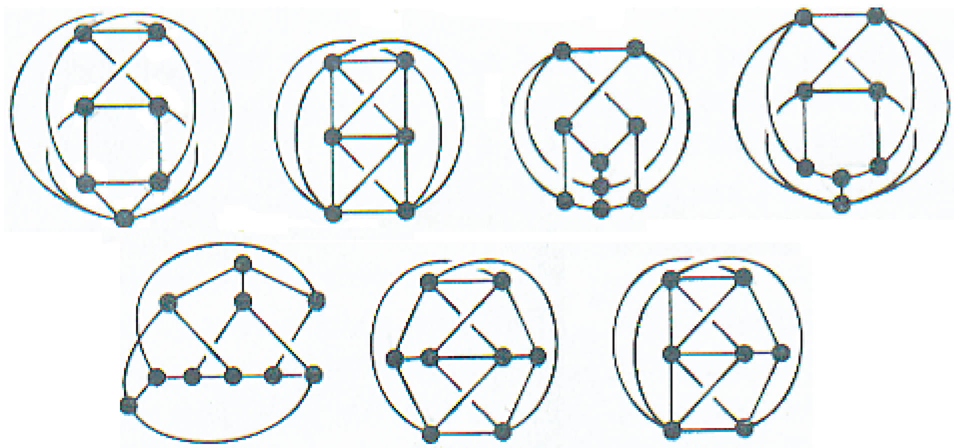

Figure 1.17: The Seven Petersen Graphs

constitute the other part. Notice that the rightmost and leftmost vertices are in different parts, yet they are not connected by an edge.

In the same way that some graphs have a link in every embedding, some graphs contain a knot in every embedding.

Definition 38 *An intrinsically knotted graph is one for which every embedding is knotted.*

One such graph is K_7. The proof that K_7 is intrinsically knotted is more complicated than the proof that K_6 is intrinsically linked; however, the idea is very similar. This was first proved by Conway and Gordon in the same paper where they proved that K_6 is intrinsically linked [CG83].

Chapter 2

Defining the Problem

In this chapter, I will discuss the motivation for the work I have done and describe the concept of my proof.

2.1 Motivation

I began this project in the summer of 2003 during my first research experience. I began by looking up unsolved questions in The Knot Book by Colin Adams[A94]. There were several that I explored briefly, however, one really caught my attention. It was Unsolved Question 3 from page 231, "Is it true that if G is intrinsically knotted, and any one vertex and the edges coming into it are removed, the remaining graph is intrinsically linked?"

One thing that attracted me to this problem is the fact that it used ideas from both graph theory and knot theory. Combining multiple disciplines of mathematics can provide very interesting results, and I feel this problem has that kind of potential.

To get some perspective on this problem, it is helpful to think of it in a slightly different way. It has been shown that intrinsic knotting is a stronger condition than intrinsic linking (lemma 10 below). That is, we know every intrinsically knotted graph is also intrinsically linked. In essence, Adams' question is asking if intrinsic knotting is so much stronger than intrinsic linking that we can actually remove a vertex, along with all edges incident to it, from an intrinsically knotted graph and still have the result be an intrinsically linked graph.

2.2 Not True in General

After working with this question for several months, we discovered a paper by Joel Foisy [F03] providing a counter example. His graph, shown in figure 2.1, is intrinsically knotted, yet removing vertex v results in a graph which is not intrinsically linked. This counterexample shows that the

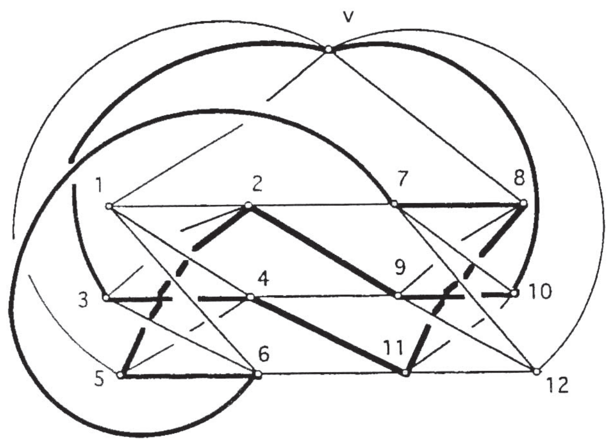

Figure 2.1: Foisy's Counterexample

conjecture is not true in general. However, the question is still interesting and still open for various classes of graphs. I think it would be strange for Foisy's counterexample to be the only one, but at the current time it is the only minor-minimal one known to exist. In searching for classes of graphs for which this conjecture holds true, we might find some other counterexamples.

We begin with a simple class of graphs, the complete partite graphs. Once we've finished with complete partite graphs, we could next consider graphs that are complete partite with one edge removed, then two edges removed and so on. To this point, I have proven the conjecture for complete partite graphs, and I believe that if one were to explore further, patterns would emerge that would simplify the problem greatly, and perhaps allow us to solve the problem in general, thus providing us a way to find all graphs for which the conjecture holds. Foisy's counterexample graph is 18 edges away from being a complete tri-partite graph. So somewhere between complete, and 18 edges away from complete, something very interesting must happen that causes the conjecture to fail.

2.3 Complete Partite Graphs

My proof of the conjecture for complete partite graphs begins by first categorizing all complete partite graphs with respect to intrinsic linking and intrinsic knotting. Although complete partite graphs are an infinite collection of graphs, we can classify them all with respect to intrinsic linking and knotting by use of this fairly intuitive lemma.

Lemma 1 *Any expansion of an intrinsically linked (respectively knotted) graph will also be intrinsically linked (respectively knotted)[MRS98].*

Proof: (Sketch)

The proof of this lemma follows from the definition of expansion. Since an expansion will never destroy a closed cycle, and we know our initial graph contains a link (respectively knot) in every embedding, the graph resulting from the expansion will also contain that same link (respectively knot) and will therefore be intrinsically linked (respectively knotted).□

The contrapositive of this lemma is also very useful, it states that any minor of a graph which is not intrinsically linked (respectively knotted) will also not be intrinsically linked (respectively knotted).

Notice that adding one vertex to any part of a complete partite graph yields an expansion of that graph, for example, $K_{6,5}$ is an expansion of $K_{5,5}$. Similarly, removing a vertex from a part results in a minor of that graph, for example, $K_{3,2,1}$ is a minor of $K_{3,3,1}$. Therefore, with this lemma in hand, it is possible to create a finite list of minor minimal complete partite graphs with respect to intrinsic linking and intrinsic knotting, and a finite list of maximal graphs which are not intrinsically linked or intrinsically knotted.

Using these lists, we check each intrinsically knotted graph to see if the conjecture holds true for that graph. To do this, consider each graph in the intrinsically knotted list, remove a vertex in every possible way, and see if each resulting graph is intrinsically linked.

Chapter 3

Proof

In this chapter I will state and prove my theorem. In section 1 I will provide all of the lemmas necessary for my proof. In sections 2 and 3 I will categorize all complete partite graphs with respect to intrinsic linking and intrinsic knotting respectively. In section 4 I will summarize sections 2 and 3 in 2 charts. And finally, in section 5 I will state and prove my theorem.

3.1 Lemmas

This section will begin with several lemmas, some of which are fairly technical. Each lemma is important; however, some require a topological background to understand the proof. I recommend all readers examine the statement of each lemma; however, those with little to no topological background should feel free to skip the proofs.

I will begin with several well known theorems, which I will state without proof. I have followed Rolfsen [R90] for much of this section. Especially the beginning of this section will look intimidating to someone who hasn't taken topology. I advise such a reader to skip ahead to the lemmas as the ideas of these theorems are fairly intuitive perhaps coming back as necessary.

Theorem 3 (Jordan Curve Theorem) *If J is a simple closed curve in $\mathbb{R}^2$, then $\mathbb{R}^2$ - J has two components, and J is the boundary of each.*

Theorem 4 (The Schönflies Theorem) *If J is a simple closed curve in $\mathbb{R}^2$, then the closure of one of the components of $\mathbb{R}^2 - J$ is homeomorphic with the unit disk.*

Remark: We'll refer to the disk as the "inside" of the curve.

Theorem 5 (Kuratowski's Reduction Theorem) *A graph G is non planar iff G is K_5 or $K_{3,3}$, or if K_5 or $K_{3,3}$ is a minor of G.*

Definition 39 *A set A in a topological space X is closed if $X\backslash A$ is open.*

Definition 40 *$K \subset X$ is compact if every open cover of K has a finite subcover. That is, if K $\subset \bigcup_{\alpha \in A} U_\alpha$ with $U_\alpha \in \Omega_X$, then $\exists \alpha_1, \alpha_2, ..., \alpha_n$ such that $K \subset \bigcup_{i=1}^{n} U_{\alpha_i}$.*

Definition 41 *$A \subset \mathbb{R}^n$ is bounded if $\exists \overrightarrow{x}, r$ such that $A \subset B(\overrightarrow{x};r)$.*

The proofs of lemmas 2, 3, and 4 below are all standard in an introductory topology class.

Lemma 2 *In $\mathbb{R}^n$ K is compact iff K is closed and bounded.*

Lemma 3 *The continuous image of a compact set is compact.*

Remark: By lemma 3, arcs and images of a graph embedding are compact sets. Since knots and links can be thought of as images of continuous maps of S^1's, they are also compact.

Definition 42 *The distance between sets A and B in $\mathbb{R}^n$ is the greatest lower bound of $\{d(p,q) |\ p \in A$ and $q \in B\}$ where d is the usual Euclidean distance in $\mathbb{R}^n$.*

Lemma 4 *The distance between two disjoint, non-empty, compact sets in $\mathbb{R}^n$ is a finite positive number.*

Lemma 5 *Any knot that bounds a disk is equivalent to the unknot.*

Strategy: Describe an isotopy from the given knot to the unknot. This isotopy will be broken down into three steps. An example of the isotopy is illustrated in figure 3.1.

Proof: (sketch)

Call the given knot K, the disk bounded by that knot D, and let x be an arbitrary point in the interior of D. Draw an arc from x to the origin. By lemma 4, there exists some $r \in \mathbb{R}^+$ such that $r < 1$ and r is less than the distance between x and K. Consider a tube of radius r about the arc from x to the origin. It's intuitive, but hard to show, that the boundaries of the disks of radius r about x and about the origin are equivalent to K and the unknot respectively. Also, we can easily describe an isotopy from the disk of radius r about x to a disk of radius r about the origin by "flowing" along the tube. Therefore, we can follow an isotopy from K, to the disk of radius r about x, to a disk of radius r about the origin, to the unlink. So we have described an isotopy that shows K is equivalent to the unknot. □

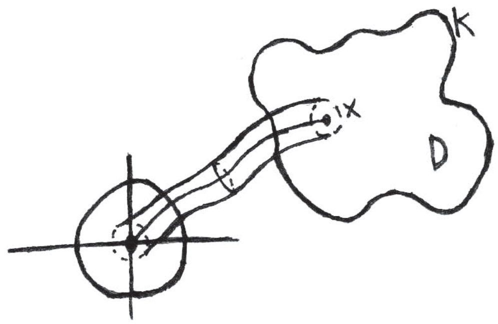

Figure 3.1: Example of Isotopy in Lemma 5

Lemma 6 *Any link whose components bound disjoint disks is equivalent to the unlink.*

Strategy: Describe an isotopy from the given link to the unlink. This isotopy will be broken down into three steps. The important point is that in every step, the two disks do not “interfere” with one another. An example of the isotopy is illustrated in figure 3.2.

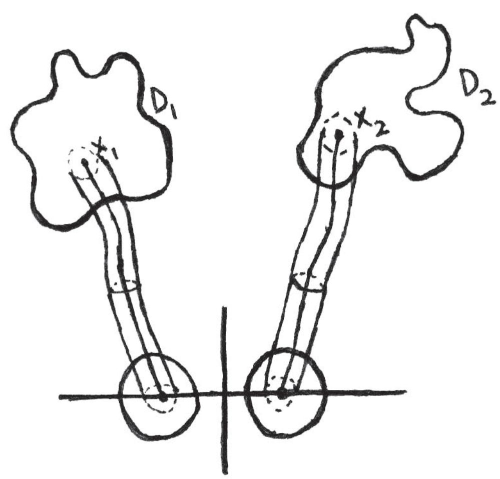

Figure 3.2: Example of Isotopy in Lemma 6

Proof: (sketch)

Call the 2 disks bounded by components of the link D_1 and D_2 with point x_1 and x_2 on the interiors of D_1 and D_2 respectively. Draw disjoint arcs from x_1 and x_2 to (-1,0,0) and (1,0,0) respectively. Since the arcs are disjoint, there must be some $r \in \mathbb{R}^+$ such that $r < \frac{1}{2}$ and a tube of radius r can surround each arc without any intersection between the two tubes (this is intuitively clear, but technically difficult to prove). The boundaries of the disks of radius r about x_1 and x_2 are equivalent to the boundaries of the disks D_1 and D_2 respectively. Similarly, the boundaries of the disks of radius r about (-1,0,0) and (1,0,0) are equivalent to their respective components of the unlink. Also, since the tube is of uniform radius r, we can easily describe an isotopy from each disk's boundary to its respective component of the unlink by simply "flowing" along the tube. Therefore, we have described a sequence of isotopies which shows a link is equivalent to the unlink (of 2 components) if its components bound disjoint disks. □

Lemma 7 *Any two-component link completely in the plane is equivalent to the unlink (of two components).*

Proof:

Notice that each component of the link is completely in a plane. Therefore, by the Schönflies Theorem, they each bound a disk. Also notice that the two components of the link cannot cross one another since they are in a plane. Therefore, there are two cases. Examples of the two cases are shown in figure 3.3.

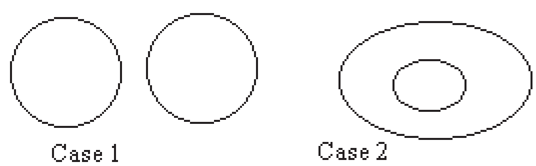

Figure 3.3: Cases of lemma 7

Case 1:

Neither component of the link is inside the other. In this case, the two components bound disjoint disks. Therefore, by lemma 6, we have the unlink.

Case 2:

One component is inside the other. Since there is no intersection between the two, there is a small annulus around the inner loop that doesn't include any of the outer loop (again, this is intuitively clear, yet difficult to prove). Isotope space so that the inner loop moves above the plane into a parallel plane, and everything in the plane outside the annulus is fixed. Now we have both components of the link on separate parallel planes. Since they are each in a plane, they each bound a disk by the Schönflies Theorem. Since they are on separate parallel planes, the two disks are disjoint. So by lemma 6, we have the unlink. □

Lemma 8 *For any two arcs A and B in a plane with common end points x and y, there exists an isotopy h_1: $\mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ such that $h_1(*,0)$ = identity and $h_1(A,1) = B$ with $h_1(x,t) = x$ and $h_1(y,t) = y$ $\forall t \in 0 \leq t \leq 1$. Moreover, given any neighborhood N of the closure of $A \cup B$, h_1 can be chosen so that $h_1(z,t) = z$ $\forall$ t, and $\forall$ points z outside N.*

Proof: (Sketch)

Case 1) Arcs A and B have no points of intersection aside from x and y.

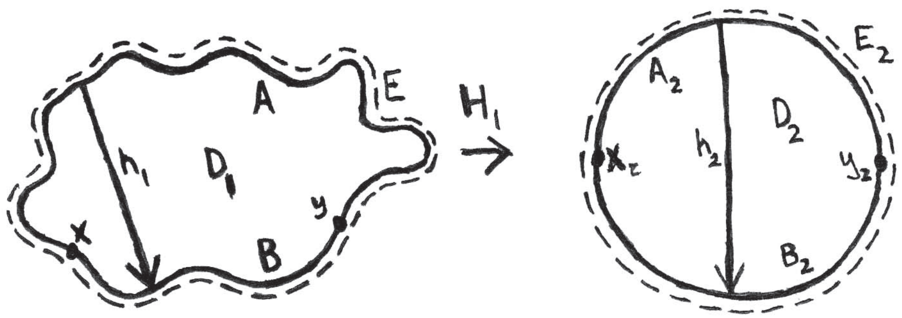

Figure 3.4: Description of Isotopy for Case 1

Figure 3.4 is an example of what is described below.

Notice that $A \cup B$ is a simple closed curve in the plane, so it bounds a disk, D_1, by the Schönflies Theorem. Also consider the simple closed curve E where E is disjoint from D_1 and $\forall e \in$E, d(e,$A \cup B$)=ϵ, for a sufficiently small $\epsilon \in \mathbb{R}^+$ (such an ϵ exists, however, it is difficult to prove that it exists). It follows intuitively, yet it is hard to prove that there exists a homeomorphism H_1: $\mathbb{R}^2 \to \mathbb{R}^2$ such that $H_1(D_1) = D_2$ and $H_1(x)$ and $H_1(y)$ are antipodal. We define A_2, B_2, E_2, x_2, and y_2 such that $A_2 = H_1(A)$, $B_2 = H_1(B)$, $E_2 = H_1(E)$, $x_2 = H_1(x)$, and $y_2 = H_1(y)$. Let ϵ_2=min{glb{d(z,$A_2 \cup B_2$)| z $\in E_2$},1}. Now there is an isotopy, h_2: $\mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ such that $h_2(A_2, 1) = B_2$, $h_2(B_2, 1)$ is inside the interior of E_2, and $h_2(z,t) = z$ for z outside E_2. Namely, beginning at any point z on A_2, isotope the plane so that z goes to B_2 along a path perpendicular to the line connecting x_2 and y_2. Consider any point $b_2 \in B_2$ and the line perpendicular to the line connecting x_2 and y_2 and containing b_2. That line will intersect A_2 at one point, call it a_2. This isotopy will move b_2 along that line, away from A_2 a distance of $\frac{\epsilon_2^2}{4*d(a_2,b_2)}$ if d(a_2,b_2) $\geq \frac{\epsilon_2}{2}$ and d(a_2,b_2) if d(a_2,b_2) $\leq \frac{\epsilon_2}{2}$. Now the isotopy is defined on A_2 and B_2, we can easily extend that to the entire plane. Notice that for every point in $b \in B_2$, $h_2(b$,1) is inside the interior of E_2. Define an isotopy of the plane $h_1 = {H_1}^{-1} \circ h_2 \circ H_1$, notice that $h_1(A, 1) = B$ and that h_1(B,1) is inside the interior of E. Also

h_1(z,t)=z for z outside E. Given any neighborhood N of A∪B, we can choose ϵ so E is inside N.

Case 2) Arcs A and B have a finite number n points of intersection. Label the points of intersection, beginning with the one closest to x along A $x_1, x_2, ..., x_n$. Label the curve along A between x and x_1 A_1, label the curve along A between x_{i-1} and x_i A_i, and label the curve along A between x_n and y A_{n+1}. In the same way, along curve B, name segments $B_1, B_2, ..., B_{n+1}$. Each A_j is isotopic to each B_j with endpoints fixed by case 1, so A is equivalent to B.

Case 3) Arcs A and B have an infinite number of points of intersection. Since we are only considering tame embeddings, we can first isotope to produce a finite number of intersections. Now we can apply case 2, and we are done. □

Notation: K_n + G is an operation on a graph G, sometimes called suspension, which adds n vertices that are all connected to one another and to every vertex in G.

Lemma 9 *K_1 + G is intrinsically linked iff G is non-planar. (first proved by Sachs [S84]).*

Proof:

(⟹)

Assume K_1 + G is intrinsically linked. For a contradiction, assume G is planar. Consider the embedding where G is in a plane, K_1 is above the plane, and each edge connecting K_1 to the plane is a straight line. Notice that any 2 cycles that are completely in G are not linked by lemma 7. For the remaining links, one component, A, must contain K_1, while the remaining component, B, must be completely in the plane of G. In this case the component A containing the K_1 vertex comes down into the plane, follows some path A_G in the plane, and then comes back up to the K_1 point. By lemma 8, there exists an isotopy, h, of the plane of G such that h(A_G,1) is a straight line in the plane of G. Moreover, h can be extended to an isotopy of $\mathbb{R}^3$ that fixes the edges of A that are incident to K_1. Since the two edges of A that are above the plane of G are straight lines, and h(A_G,1) is a straight line, the pieces of h(A,1) form a triangle, and therefore are in a plane P. Since h(A_G,1) and h(B,1) are images of cycles in a graph, they must be disjoint compact sets by lemma 3. By lemma 4 the distance between h(A_G,1) and h(B,1) is d∈ $\mathbb{R}^+$. So, if we construct a rectangle in the plane of G around h(A_G,1) so that the distance from each point on the rectangle to h(A_G,1) is less than or equal to $\frac{d}{2}$, the interior of that rectangle will not intersect h(B,1). There exists an isotopy of $\mathbb{R}^3$ that will move h(A_G,1) above the plane of G and parallel to plane P, while leaving any points in the plane of G outside the rectangle unchanged. The result is one component completely in the plane of G, one component completely in plane P and above the plane of G. Therefore the disks bounded by the two components are disjoint. By lemma 6, this link is equivalent to the unlink. It follows that there are no non-trivial links in this embedding of the graph, a contradiction. Therefore, G must be non-planar.

($\Longleftarrow$)

Assume G is non-planar. By Kuratowski's reduction theorem, G must contain $K_{3,3}$ or K_5 as a minor.

Case 1) G has $K_{3,3}$ as a minor, so G + K_1 has $K_{3,3,1}$ as a minor. $K_{3,3,1}$ is one of the graphs pictured in figure 1.17 and is therefore intrinsically linked by theorem 2. So by lemma 1, G + K_1 is intrinsically linked.

Case 2) G has K_5 as a minor, then G + K_1 has K_6 as a minor which is intrinsically linked by theorem 1, so by lemma 1, G + K_1 is intrinsically linked. $\square$

Lemma 10 *Any graph that is intrinsically knotted is also intrinsically linked.*

This was proven by Robertson, Seymour, and Thomas [RST95]. I will omit the proof as it is about 40 pages long. A summary is available in [RST93].

Lemma 11 *K_2 + G is intrinsically knotted iff G is non-planar (Theorem 2.1 in [Fl]).*

Remark: Notice that adding K_2 to a partite graph will add 2 parts, each with one vertex. For example, $K_{2,2}$ + K_2 is $K_{2,2,1,1}$.

Proof:

($\Longrightarrow$) Assume K_2 + G is intrinsically knotted. For a contradiction, assume G is planar. Call the 2 vertices of K_2 a and b. Consider an embedding with G in a plane, a above the plane, b below the plane, and each edge connecting a or b to a vertex in the plane a straight line. Also, notice that edge (a, b) must pass through the plane of G somewhere. By lemma 3, G is compact, and so, by lemma 2, it is bounded in the plane. Assume that in our embedding, edge (a, b) passes through the plane of G at exactly one point c which is outside of a bounded region containing G. Also assume the part of edge (a, b) between a and c is a straight line, and the part of edge (a, b) between b and c is also a straight line. I will show that no cycle within this graph is knotted. There will be 4 cases: 1) a cycle completely in the plane, 2) a cycle using exactly one of a or b, 3) a cycle using a and b, that doesn't use edge (a, b), and 4) a cycle using a and b and edge (a, b).

Case 1) Any cycle completely in a plane is a simple closed curve. So by the Schönflies Theorem, it is the boundary of a disk, and is therefore the unknot by lemma 5.

Case 2) In this case, our cycle uses exactly 1 of the vertices a or b, the rest of the vertices of the cycle are in a plane. By lemma 8 there exists an isotopy of the plane of G such that the path of our cycle through the plane is a straight line. This extends to an isotopy of $\mathbb{R}^3$ which sends our cycle to 3 straight lines segments. Thus, the image is in a plane so it bounds a disk by the Schönflies Theorem, and is therefore equivalent to the unknot by lemma 5.

Case 3) Strategy: we will describe an isotopy that will move our cycle into a plane.

A cycle in this case uses vertices a and b without using edge (a, b). In the cycle vertex a is connected to 2 vertices in the plane, call them a_1 and a_2. Similarly, b is connected to 2 vertices in the plane, call them b_1 and b_2, where a_1 and a_2 are connected to b_1 and b_2 respectively through a series of edges in the plane or $a_1 = b_1$ (or respectively, a_2=b_2). Let γ_{b_1,a_1} and γ_{b_2,a_2} be our arcs from b_1 to a_1 and b_2 to a_2 respectively. Since γ_{b_1,a_1} and γ_{b_2,a_2} are not closed paths, plane$\backslash(\gamma_{b_1,a_1} \cup \gamma_{b_2,a_2})$ is connected. Therefore, there exists an arc γ_{a_1,a_2} in the plane disjoint from γ_{b_1,a_1} and γ_{b_2,a_2} except for endpoints. It follows that $\gamma_{b_1,a_1} \cup \gamma_{a_1,a_2} \cup \gamma_{a_2,b_2}$ is a non-self-intersecting arc which is not closed. Since this arc is not closed, plane$\backslash(\gamma_{b_1,a_1} \cup \gamma_{a_1,a_2} \cup \gamma_{a_2,b_2})$ is connected, so there exists an arc γ_{b_2,b_1} that only intersects $\gamma_{b_1,a_1} \cup \gamma_{a_1,a_2} \cup \gamma_{a_2,b_2}$ at endpoints. Therefore, $\gamma_{b_1,a_1} \cup \gamma_{a_1,a_2} \cup \gamma_{a_2,b_2} \cup \gamma_{b_2,b_1}$ is a simple closed curve in the plane, therefore it bounds a disk by Schönflies Theorem, so by lemma 5, it is the unknot.

By lemma 8, there exists an isotopy h_1 of the plane of G such that $h_1(\gamma_{b_1,b_2},1)$ is a straight line. Moreover, h_1 can be extended to an isotopy of $\mathbb{R}^3$ such that edges (b, b_1) and (b, b_2) remain straight. Since edge (b, b_1) and edge (b, b_2) are straight lines, they describe a triangle and are therefore in a plane; call it B. So by lemma 8, there exists a homeomorphism of plane B such that $h_2((b_1, b) \cup (b, b_2),1)$ is in the plane of G. Similarly, we can describe a homeomorphism h_3 such that $h_3((a_1, a) \cup (a, a_2),1)$ is in the plane of G. Note, there is a possibility that B will meet edges (a, a_1) or (a, a_2) so they're no longer straight lines after h_2. We can avoid this by first isotoping $(a, a_1) \cup (a, a_2)$ into a plane parallel to B.

Now, if we in tern apply h_1, h_2, and h_3, our cycle will be completely in the plane of G. So by the Schönflies Theorem, our cycle bounds a disk; therefore by lemma 5, it is the unknot.

Case 4) Notice that the edge (a, b) must pass through the plane of G. Recall that the point of intersection is c. Now if we consider the argument for case 3, and let c be γ_{b_2,a_2}, then case 4 reduces to case 3 and is therefore proven.

($\Longleftarrow$) Assume G is non-planar. By Kuratowski's reduction theorem, G must contain $K_{3,3}$ or K_5 as a minor.

Case 1) G has $K_{3,3}$ as a minor, so G + K_2 has $K_{3,3,1,1}$ as a minor, $K_{3,3,1,1}$ was shown to be intrinsically knotted in [F02], so by lemma 1, G + K_2 is intrinsically knotted.

Case 2) G has K_5 as a minor, so G + K_2 has K_7 as a minor, K_7 was shown to be intrinsically knotted in [CG83], so by lemma 1, G + K_2 is intrinsically knotted. □

Lemma 12 *$K_{n_1+n_2,n_3,...,n_i}$ is a minor of $K_{n_1,n_2,n_3,...,n_i}$.*

Recall that although we usually write the parts in decending order, we can write them in any order we like. Therefore, this lemma allows us to "combine" any two parts of a complete partite graph to get a minor of that graph. For example, by this lemma $K_{3,4,3}$ is a minor of $K_{2,1,4,3}$, but we usually write parts in decending order, so it is more natural to say $K_{4,3,3}$ is a minor of $K_{4,3,2,1}$.

Proof:

In $K_{n_1,n_2,n_3,...,n_i}$, delete each edge between any vertex in n_1 and any vertex in n_2. (By abuse of notation, let n_i denote the set of vertices in the i^{th} part.) Now each vertex in $n_1 \cup n_2$ is connected to each vertex from n_3,...,n_i, and no two vertices in $n_1 \cup n_2$ are connected to one another. By definition, this is $K_{n_1+n_2,n_3,...,n_i}$. Therefore $K_{n_1+n_2,n_3,...,n_i}$ is a minor of $K_{n_1,n_2,n_3,...,n_i}$.□

3.2 Categorizing with Respect to Intrinsic Linking

In this section, I will categorize all complete partite graphs with respect to intrinsic linking. I will categorize them according to how many parts they have. I will also consider complete graphs. I will begin by categorizing complete graphs, then bipartite graphs, then tripartite graphs, and so on. This classification is an original contribution to the literature. At the beginning of each subsection, I identify minimal intrinsically linked graphs and maximal graphs that are not intrinsically linked.

3.2.1 Complete Graphs

I will show that K_6 is intrinsically linked and K_5 is not intrinsically linked. Notice that all complete graphs, K_n are covered here. If n>6 K_6 is a minor of K_n so K_n is intrinsically linked by lemma 1. Conversely, if n<5, K_n is a minor of K_5 so K_n is not intrinsically linked by lemma 1.

First notice that any closed cycle requires three vertices, and a link requires at least two closed cycles. Therefore, for a graph to be intrinsically linked, it must have at least 6 vertices. It follows that K_5 is not intrinsically linked.

By Theorem 1, K_6 is intrinsically linked.

3.2.2 Bipartite Graphs

I will show that $K_{4,4}$ is intrinsically linked, and $K_{3,3}$ is not intrinsically linked. Notice that any cycle in a bipartite graph must use an equal number of vertices from each part. Therefore, any graph of the form $K_{m,n}$ with m $\geq$ n will behave exactly like $K_{n,n}$ with respect to intrinsic linking (and intrinsic knotting). So only graphs of the form $K_{n,n}$ need to be considered. If n > 4, $K_{n,n}$ contains $K_{4,4}$ and is intrinsically linked. If n < 3, $K_{n,n}$ is a minor of $K_{3,3}$ and is not intrinsically linked.

Also, any cycle in a bipartite graph must include at least two vertices from each part. To see this, consider beginning at vertex a_1 in part A. The first edge of the path will be to vertex b_1 in part B. Our next edge will be to vertex a_2 in part A which is necessarily different from a_1. There is no edge between vertices a_1 and a_2 since they are in the same part, so to get to a_1 from a_2 we must first travel to another vertex in B. We now follow an edge from a_2 to b_2 which is necessarily different from b_1, and finally we can follow an edge from b_2 to a_1 or we can continue further to get a cycle of more than 4 vertices.

Since each cycle requires at least 2 vertices from each part, and a link requires 2 cycles, for intrinsical linking, a graph will require at least 4 vertices in each part, so $K_{3,3}$ is not intrinsically linked.

$K_{4,4}$-{e} is a Petersen graph, as shown in figure 1.17, so by theorem 2 and lemma 1, $K_{4,4}$ is intrinsically linked.

3.2.3 Tripartite Graphs

I will show that $K_{3,3,1}$ and $K_{4,2,2}$ are intrinsically linked, and $K_{3,2,2}$ and $K_{n,2,1}$ are not intrinsically linked. Note that if $K_{a,b,c}$ is not one of these 4 graphs, it is either a minor of $K_{3,2,2}$ or $K_{n,2,1}$, or else it contains $K_{3,3,1}$ or $K_{4,2,2}$ as a minor. Although I will no longer state it, this idea will carry through in each characterization section.

As shown in figure 3.5, $K_{n,2}$ is planar, so by lemma 9, $K_{n,2,1}$ is not intrinsically linked.

As shown in figure 3.5, $K_{2,2,2}$ is planar, so by lemma 9, $K_{2,2,2,1}$ is not intrinsically linked. By lemma 12, $K_{3,2,2}$ is a minor of $K_{2,2,2,1}$. So by lemma 1, $K_{3,2,2}$ is not intrinsically linked.

$K_{3,3,1}$ is a Petersen graph as shown in figure 1.17, so it is intrinsically linked by theorem 2. In subsection 3.2.2, we showed that $K_{4,4}$ is intrinsically linked. By lemma 12, $K_{4,4}$ is a minor of $K_{4,2,2}$, so by lemma 1, $K_{4,2,2}$ is intrinsically linked.

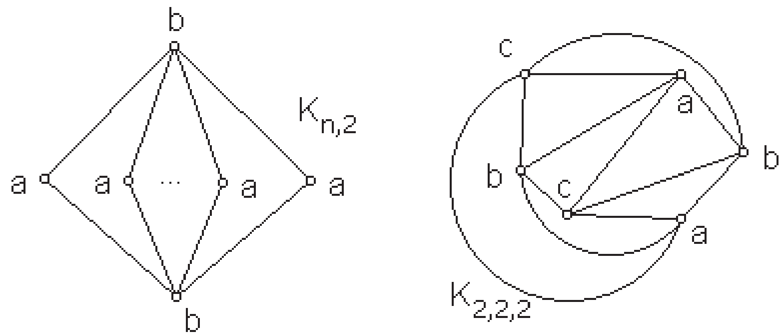

Figure 3.5: Planar Embeddings of $K_{n,2}$ and $K_{2,2,2}$

3.2.4 4 partite graphs

I will show that $K_{2,2,2,2}$ and $K_{3,2,1,1}$ are intrinsically linked, and $K_{2,2,2,1}$ and $K_{n,1,1,1}$ are not intrinsically linked.

As shown in figure 3.5, $K_{2,2,2}$ is planar, so by lemma 9, $K_{2,2,2,1}$ is not intrinsically linked.

As illustrated in figure 3.6, $K_{n,1,1}$ is planar, so by lemma 9, $K_{n,1,1,1}$ is not intrinsically linked.

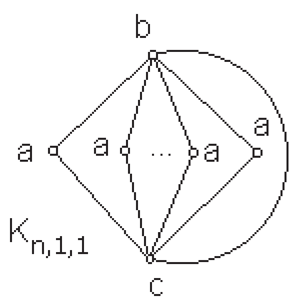

Figure 3.6: A Planar Embedding of $K_{n,1,1}$

In subsection 3.2.3, we showed that $K_{4,2,2}$ is intrinsically linked. By lemma 12, $K_{4,2,2}$ is a minor of $K_{2,2,2,2}$, so by lemma 1, $K_{2,2,2,2}$ is intrinsically linked.

Similarly, in subsection 3.2.3, we showed that $K_{3,3,1}$ is intrinsically linked. By lemma 12, $K_{3,3,1}$ is a minor of $K_{3,2,1,1}$, so by lemma 1, $K_{3,2,1,1}$ is intrinsically linked.

3.2.5 5 partite graphs

I will show that $K_{2,2,1,1,1}$ and $K_{3,1,1,1,1}$ are intrinsically linked and $K_{2,1,1,1,1}$ is not intrinsically linked.

As illustrated in figure 3.7, $K_{2,1,1,1}$ is planar, so by lemma 9 $K_{2,1,1,1,1}$ is not intrinsically linked.

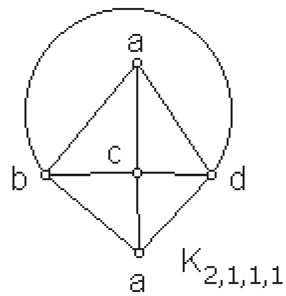

Figure 3.7: A Planar Embedding $K_{2,1,1,1}$

In subsection 3.2.4 we showed that $K_{3,2,1,1}$ is intrinsically linked. By lemma 12, $K_{3,2,1,1}$ is a minor of both $K_{2,2,1,1,1}$ and $K_{3,1,1,1,1}$. So by lemma 1, $K_{2,2,1,1,1}$ and $K_{3,1,1,1,1}$ are both intrinsically linked.

3.2.6 k Partite Graphs (k $\geq$ 6)

Every complete partite graph with six or more parts is intrinsically linked.

Every complete partite graph with six or more parts will either be $K_{1,1,1,1,1,1}$, or have it as a minor. But $K_{1,1,1,1,1,1}$ is six vertices all connected to one another, which is also called K_6. We have shown that K_6 is intrinsically linked in subsection 3.2.1, so by lemma 1, any complete partite graph with 6 or more parts is intrinsically linked.

3.3 Categorizing with Respect to Intrinsic Knotting

In this section I will categorize all complete partite graphs with respect to intrinsic knotting in the same manner as I categorized them with respect to intrinsic linking in the previous section. Although Flemming was the first to do this [Fl], we independently came up with the same results.

Rather than follow his proof, I present our own argument below. In each subsection, we present minimal intrinsically knotted examples and maximal examples that are not intrinsically knotted.

3.3.1 Complete Graphs

I will show that K_7 is intrinsically knotted and K_6 is not intrinsically knotted.

As shown in figure 3.8, K_4 is planar, so by lemma 11, K_6 is not intrinsically knotted.

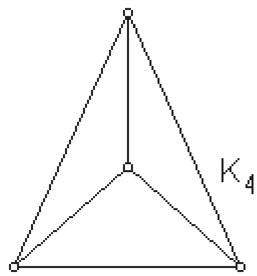

Figure 3.8: A Planar Embedding of K_4

By Kuratowski's Reduction Theorem, K_5 is non-planar, so by lemma 11, K_7 is intrinsically knotted. (This was first proven by Conway and Gordon [CG83]).

3.3.2 Bipartite Graphs

As was shown in [S88], $K_{5,5}$ is intrinsically knotted. I will show that $K_{4,4}$ is not intrinsically knotted.

Recall from section 3.2.2 that we need only consider bipartite graphs of the form $K_{n,n}$.

As shown in section 3.2.3, $K_{n,2}$ is planar, so $K_{4,2}$ is planar. So by lemma 11, $K_{4,2,1,1}$ is not intrinsically knotted. Therefore, by lemmas 12 and 1, $K_{4,4}$ is not intrinsically knotted.

3.3.3 Tripartite Graphs

I will show that $K_{3,3,3}$, $K_{4,3,2}$, and $K_{4,4,1}$ are intrinsically knotted, and that $K_{3,3,2}$, $K_{n,2,2}$ and $K_{n,3,1}$ are not intrinsically knotted.

$K_{3,3,1,1}$ is minor minimal with respect to intrinsic knotting [F02]. By lemma 12 $K_{3,3,2}$ is a minor of $K_{3,3,1,1}$, so by the definition of minor minimal, $K_{3,3,2}$ is not intrinsically knotted.

$K_{n,2}$ is planar, as shown in figure 3.5. So $K_{n,2,1,1}$ is not intrinsically knotted by lemma 11. So by lemmas 12 and 1, $K_{n,2,2}$ is not intrinsically knotted.

$K_{n,1,1}$ is planar, as pictured in subsection 3.2.4. So $K_{n,1,1,1,1}$ is not intrinsically knotted by lemma 11. So by lemmas 12 and 1, $K_{n,3,1}$ is not intrinsically knotted.

Next I will show that $K_{3,3,3}$ is intrinsically knotted by showing that a minor of it is intrinsically knotted.

The first graph in figure 3.9, labeled H_9 is shown to be intrinsically knotted in [KS92]. Add the dashed edges in the second picture to arrive at the resulting graph, which is seen to be $K_{3,3,3}$ in the third picture. We have shown that H_9 is a minor of $K_{3,3,3}$, so by lemma 1, $K_{3,3,3}$ is intrinsically knotted.

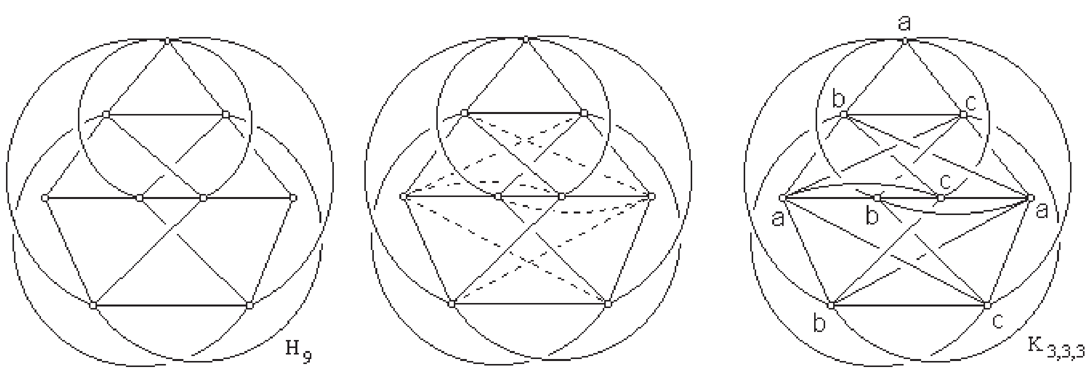

Figure 3.9: H_9 is a Minor of $K_{3,3,3}$

Similarly, I will now show that $K_{4,3,2}$ is intrinsically knotted by showing it has $K_{3,3,1,1}$ as a minor.

As stated above, $K_{3,3,1,1}$ is intrinsically knotted [F02]. As illustrated in figure 3.10, $K_{3,3,1,1}$ is a minor of $K_{4,3,2}$, so by lemma 1, $K_{4,3,2}$ is intrinsically knotted.

I will now show that $K_{4,4,1}$ is intrinsically knotted by showing it has $K_{3,3,1,1}$ as a minor. We begin with $K_{3,3,1,1}$ and the knowledge that it is intrinsically knotted. As illustrated in figure 3.11, we split the vertex d into vertices d_1 and d_2 such that d_1 is connected to the vertices labeled a and d_2 is connected to the vertices labeled b and c. Also recall that when you split a vertex, the resulting vertices are connected to one another, so d_1 is connected to d_2. Then, add an edge, illustrated with the dashed line, between vertices d_1 and the one previously labeled c. In the third picture, it is clear that the resultant graph is $K_{4,4,1}$. So it has been shown that $K_{3,3,1,1}$ is a minor of $K_{4,4,1}$. By lemma 1, $K_{4,4,1}$ is intrinsically knotted.

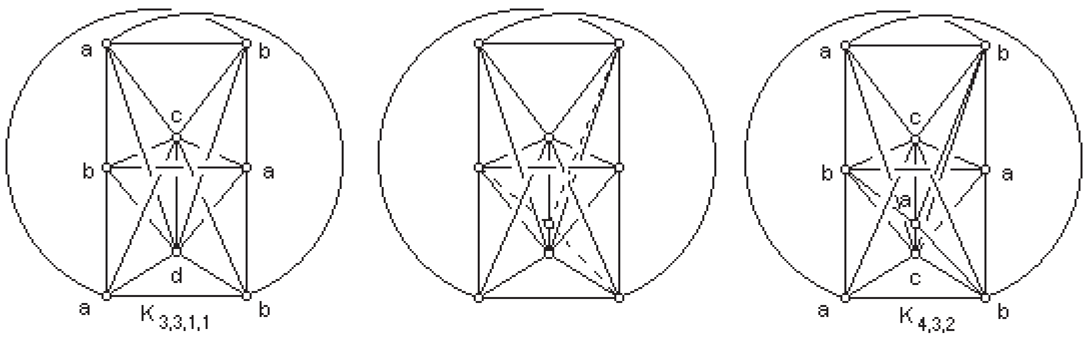

Figure 3.10: $K_{3,3,1,1}$ is a Minor of $K_{4,3,2}$

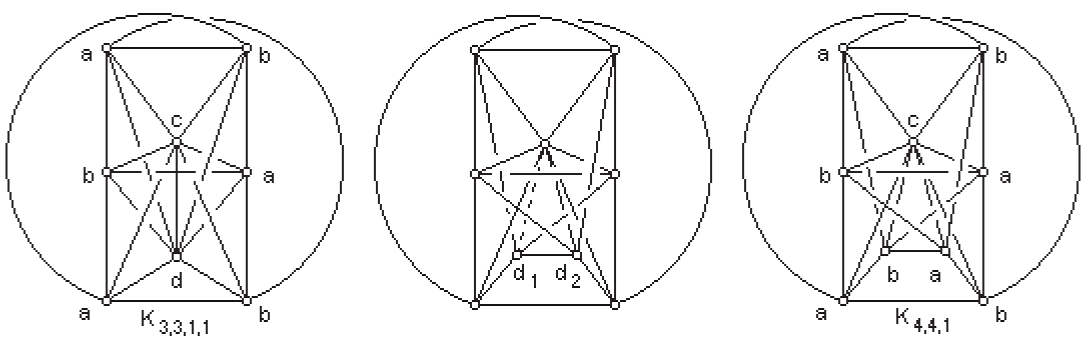

Figure 3.11: $K_{3,3,1,1}$ is a Minor of $K_{4,4,1}$

3.3.4 4 Partite Graphs

I will show that $K_{3,2,2,2}$, $K_{4,2,2,1}$, and $K_{3,3,1,1}$ are intrinsically knotted and $K_{2,2,2,2}$, $K_{3,2,2,1}$, and $K_{n,2,1,1}$ are not intrinsically knotted.

$K_{2,2,2}$ is planar as shown in figure 3.5, so by lemma 11, $K_{2,2,2,1,1}$ is not intrinsically knotted. By lemma 12, $K_{2,2,2,2}$ is a minor of $K_{2,2,2,1,1}$, so by lemma 1, $K_{2,2,2,2}$ is not intrinsically knotted.

Similarly, since $K_{2,2,2,1,1}$ is not intrinsically knotted, by lemmas 12 and 1, $K_{3,2,2,1}$ is not intrinsically knotted.

As shown in figure 3.5, $K_{n,2}$ is planar, so $K_{n,2,1,1}$ is not intrinsically knotted by lemma 11.

Recall from subsection 3.3.3 that $K_{4,3,2}$ is intrinsically knotted. By lemma 12, $K_{4,3,2}$ is a minor of $K_{4,2,2,1}$, so by lemma 1, $K_{4,2,2,1}$ is intrinsically knotted.

I will now show that $K_{3,2,2,2}$ is intrinsically knotted by showing that it has $K_{3,3,1,1}$ as a minor. As shown in figure 3.12, we start with $K_{3,3,1,1}$ which is intrinsically knotted. Split the leftmost vertex labeled b into vertices b_1 and b_2. Recall that b_1 and b_2 are connected, and each vertex that was connected to b must now be connected to either b_1 or b_2. Connect each vertex labeled a or d to b_1 and connect vertex c to b_2. Now add edges which are represented by dashed lines. The resulting graph is $K_{3,2,2,2}$. In this way, we have shown that $K_{3,3,1,1}$ is a minor of $K_{3,2,2,2}$, so by lemma 1, $K_{3,2,2,2}$ is intrinsically knotted. Note that in the accompanying picture, crossings were left out because the picture was getting cluttered. Since we are considering intrinsic knotting, which is embedding independent, it is irrelevant which edge is below and which is above.

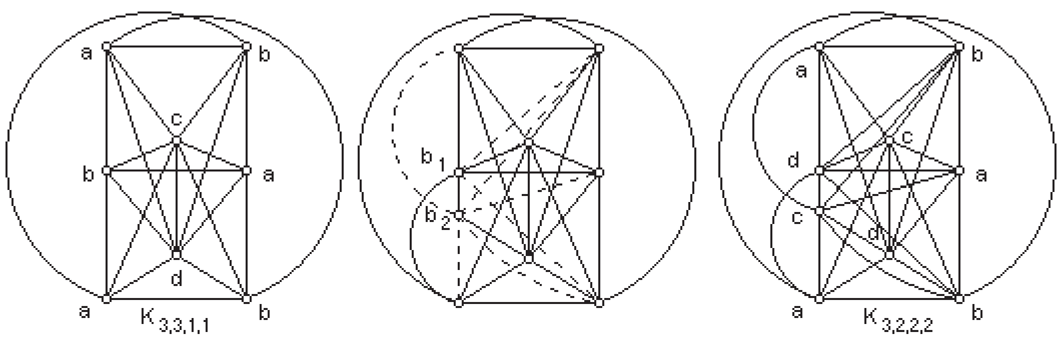

Figure 3.12: $K_{3,3,1,1}$ is a Minor of $K_{3,2,2,2}$

3.3.5 5 Partite Graphs

I will show that $K_{2,2,2,2,1}$ and $K_{3,2,1,1,1}$ are intrinsically knotted and $K_{2,2,2,1,1}$ and $K_{n,1,1,1,1}$ are not intrinsically knotted.

By figures 3.5 and 3.6, $K_{2,2,2}$ and $K_{n,1,1}$ are planar. So by lemma 11, $K_{2,2,2,1,1}$ and $K_{n,1,1,1,1}$ are not intrinsically knotted.

Recall from subsection 3.3.4 that $K_{4,2,2,1}$ is intrinsically knotted. By lemma 12, $K_{4,2,2,1}$ is a minor of $K_{2,2,2,2,1}$, so by lemma 1, $K_{2,2,2,2,1}$ is intrinsically knotted.

As mentioned many times, $K_{3,3,1,1}$ is intrinsically knotted. By lemma 12, $K_{3,3,1,1}$ is a minor of $K_{3,2,1,1,1}$, so by lemma 1, $K_{3,2,1,1,1}$ is intrinsically knotted.

3.3.6 6 Partite Graphs

I will show that $K_{2,2,1,1,1,1}$ and $K_{3,1,1,1,1,1}$ are intrinsically knotted and $K_{2,1,1,1,1,1}$ is not intrinsically knotted.

As shown in figure 3.7, $K_{2,1,1,1}$ is planar, so by lemma 11, $K_{2,1,1,1,1,1}$ is not intrinsically knotted.

$K_{2,2,1,1}$ and $K_{3,1,1,1}$ each contain $K_{3,3}$ as a minor by lemma 12, therefore, they are each non-planar by Kuratowski's Reduction Theorem. So by lemma 11, $K_{2,2,1,1,1,1}$ and $K_{3,1,1,1,1,1}$ are intrinsically knotted.

3.3.7 k Partite Graph (k $\geq$ 7)

I will show that every complete partite graph with 7 or more parts is intrinsically knotted.

Every complete partite graph with seven or more parts is $K_{1,1,1,1,1,1,1}$ or contains it as a minor. But $K_{1,1,1,1,1,1,1}$ is seven vertices all connected to one another, which is also called K_7. We have shown that K_7 is intrinsically knotted in subsection 3.3.1, so by lemma 1, any complete partite graph with 7 or more parts is intrinsically knotted.

3.4 Summarizing

All of the information in the previous two sections can be summarized into tables, one for linking and one for knotting. In each table, the top row is the number of parts, the second row is the minor minimal intrinsically linked (respectively knotted) complete partite graphs. The third row is the maximal complete partite graphs which are not intrinsically linked (respectively knotted).

In conjunction with lemma 1, these tables allow us to determine intrinsic knotting and intrinsic linking of all complete partite graphs. For example, a 3-partite graph $K_{a,b,c}$ that is not in the intrinsic linking chart will either contain $K_{3,3,1}$ or $K_{4,2,2}$ as a minor and be intrinsically linked, or it will be a minor of $K_{3,2,2}$ or $K_{n,2,1}$ and not be intrinsically linked. Similarly, $K_{a,b,c}$ that is not in the intrinsic knotting chart will either contain $K_{3,3,3}$, $K_{4,3,2}$, or $K_{4,4,1}$ as a minor and be intrinsically knotted, or it will be a minor of $K_{3,3,2}$, $K_{n,2,2}$, or $K_{n,3,1}$ and not be intrinsically knotted. For example $K_{5,2,1,1}$ contains $K_{3,2,1,1}$ and is intrinsically linked, and $K_{5,2,1,1}$ is an example of $K_{n,2,1,1}$ and is therefore not intrinsically knotted.

k	1	2	3	4	5	≥ 6
linked	6	4,4	3,3,1 4,2,2	2,2,2,2 3,2,1,1	2,2,1,1,1 3,1,1,1,1	All
not linked	5	n,3	3,2,2 n,2,1	2,2,2,1 n,1,1,1	2,1,1,1,1	None

Table 3.1: Intrinsic Linking of Complete Partite Graphs.

k	1	2	3	4	5	6	≥ 7
knotted	7	5,5	3,3,3 4,3,2 4,4,1	3,2,2,2 4,2,2,1 3,3,1,1	2,2,2,2,1 3,2,1,1,1	2,2,1,1,1,1 3,1,1,1,1,1	All
not knotted	6	4,4	3,3,2 n,2,2 n,3,1	2,2,2,2 3,2,2,1 n,2,1,1	2,2,2,1,1 n,1,1,1,1	2,1,1,1,1,1	None

Table 3.2: Intrinsic knotting of k-partite graphs.

3.5 My Theorem

Theorem 6 *If G is an intrinsically knotted complete partite graph, the removal of any vertex will result in a graph that is still intrinsically linked.*

Strategy:

I will show that for each graph which is knotted according to the second table, we can remove any one vertex and the resulting graph will either be in the list of intrinsically linked graphs, or it will contain one of those graphs as a minor. I will show this one part at a time.

Proof:

(k=1)

K_7 is intrinsically knotted, removing any vertex results in K_6, which is intrinsically linked.

(k=2)

$K_{5,5}$ is intrinsically knotted, removing any vertex results in $K_{5,4}$. Since $K_{4,4}$ is intrinsically linked, and $K_{4,4}$ is a minor of $K_{5,4}$, by lemma 1 $K_{5,4}$ is intrinsically linked.

(k=3)

$K_{3,3,3}$ is intrinsically knotted, removing any vertex results in $K_{3,3,2}$. $K_{3,3,2}$ contains $K_{3,3,1}$ as a minor which is intrinsically linked. So $K_{3,3,2}$ is intrinsically linked by lemma 1.

$K_{4,3,2}$ is intrinsically knotted, removing any vertex results in $K_{4,3,1}$, $K_{4,2,2}$, or $K_{3,3,2}$. $K_{4,2,2}$ is intrinsically linked by the intrinsically linked table, $K_{4,3,1}$ and $K_{3,3,2}$ both have $K_{3,3,1}$ as a minor which is intrinsically linked, so both are intrinsically linked by lemma 1.

$K_{4,4,1}$ is intrinsically knotted, removing any vertex results in $K_{4,4}$ or $K_{4,3,1}$. $K_{4,4}$ is in the intrinsically linked table, and $K_{4,3,1}$ contains $K_{3,3,1}$ as a minor which is intrinsically linked, so $K_{4,3,1}$ is intrinsically linked by lemma 1.

(k=4)

$K_{3,2,2,2}$ is intrinsically knotted, removing any vertex results in $K_{3,2,2,1}$ or $K_{2,2,2,2}$. $K_{2,2,2,2}$ is intrinsically linked by the table, and $K_{3,2,2,1}$ contains $K_{3,2,1,1}$ as a minor which is intrinsically linked, so by lemma 1, $K_{3,2,2,1}$ is intrinsically linked.

$K_{4,2,2,1}$ is intrinsically knotted, removing any vertex results in $K_{4,2,2}$, $K_{4,2,1,1}$, or $K_{3,2,2,1}$. $K_{4,2,2}$ is intrinsically linked according to the table, and $K_{4,2,1,1}$ and $K_{3,2,2,1}$ both contain $K_{3,2,1,1}$ as a minor which is intrinsically linked, so by lemma 1, $K_{4,2,1,1}$ and $K_{3,2,2,1}$ are intrinsically linked.

$K_{3,3,1,1}$ is intrinsically knotted, removing any vertex results in $K_{3,3,1}$ or $K_{3,2,1,1}$, both of which are intrinsically linked by the table.

(k=5)

$K_{2,2,2,2,1}$ is intrinsically knotted, removing any vertex results in $K_{2,2,2,2}$ or $K_{2,2,2,1,1}$. $K_{2,2,2,2}$ is in the intrinsically linked table, while $K_{2,2,2,1,1}$ contains $K_{2,2,1,1,1}$ as a minor which is intrinsically linked, so by lemma 1 $K_{2,2,2,1,1}$ is intrinsically linked.

$K_{3,2,1,1,1}$ is intrinsically knotted, removing any vertex results in $K_{3,2,1,1}$, $K_{3,1,1,1,1}$, or $K_{2,2,1,1,1}$, all of which are in the intrinsically linked table.

(k=6)

$K_{2,2,1,1,1,1}$ is intrinsically knotted, removing any vertex results in $K_{2,2,1,1,1}$ or $K_{2,1,1,1,1,1}$, both of which are in the intrinsically linked table.

$K_{3,1,1,1,1,1}$ is intrinsically knotted, removing any vertex results in $K_{3,1,1,1,1}$ or $K_{2,1,1,1,1,1}$, both of which are in the intrinsically linked table.

(k $\geq$ 7)

All graphs with 7 or more parts are intrinsically knotted, removing a vertex will always result in a graph with 6 or more parts, which will be intrinsically linked according to the table. $\square$

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