

Knotting of Graphs on Nine Vertices and 28 or More Edges

Jody Ryker

Abstract

We show that there is only one graph on nine vertices and 28 or more edges that is minor minimal intrinsically knotted. We present a general strategy for finding minor minimal intrinsically knotted graphs.

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Introduction

1. Summary of Results

We would like to answer the question: what types of graphs can be minor minimal intrinsically knotted (MMIK)? A graph is said to be IK if every representation of that graph is knotted. A graph is MMIK if that graph is IK and has no IK minor. We will give more complete definitions later in this chapter. We know that there are finitely many MMIK graphs [RS]. The next step is to list them all. Using work by Conway and Gordon [CG], we know that the only MMIK graph on seven vertices or less is K_7 . Due to further research ([BBFFHL, CMOPRW]), we also know that the only MMIK graphs on eight vertices are H_8 and $K_{3,3,1,1}$. The next natural question is: what graphs on nine vertices are MMIK? We know by previous work ([BM, LKLO]) that there are only 14 MMIK graphs on nine vertices and 21 or fewer edges. From a computer program created by Morris [MS], we have a list of IK graphs on nine vertices. This program was unable to determine the knotting of 32 graphs. We would like to decide which of these graphs are IK in a way that doesn't depend on computers.

In this thesis, we determine which graphs on nine vertices and 28 or more edges are MMIK. It was already known that there exists an MMIK graph on 28 edges [GMN]. Through our work, we found that this is in fact the only MMIK graph on nine vertices and 28 or more edges. Due to a useful theorem of [CMOPRW] (stated in the next section), we need only consider graphs with fewer than 31 edges. Our strategy is to look at graphs with 30, 29, and 28 edges in turn, showing that there are no MMIK graphs in the first two cases and exactly one of 28 edges.

In order to explain our results in more detail, we present some preliminary definitions and lemmas in the next section. We conclude the chapter with a section that gives a detailed description of our results and how the thesis is organized.

2. Definitions and Lemmas

Below are some helpful definitions and theorems.

DEFINITION 1. A graph G is a collection of edges $E(G)$ and vertices $V(G)$. We describe G as (v, e) where v is the number of vertices in G and e is the number of edges in G .

DEFINITION 2. A path P in a graph G is a sequence of edges and vertices with consecutive edges sharing a common vertex.

DEFINITION 3. A graph G is connected if there exists a path from any vertex in G to any other vertex in G .

DEFINITION 4. A subgraph G' of a graph G is a graph with $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$.

DEFINITION 5. A component C of a graph G is a largest connected subgraph. That is, C is a connected subgraph and there is no connected subgraph that properly contains C .

DEFINITION 6. A cycle in a graph G is a path starting and terminating on the same vertex in G .

DEFINITION 7. In a graph G , an edge contraction of edge e with adjacent vertices v_1 and v_2 is the replacement of v_1 and v_2 with a single vertex v such that the edges incident to v are the edges that were incident to either v_1 or v_2 excluding e .

DEFINITION 8. A minor of a graph G is a graph obtained through some number of edge deletions, vertex deletions, or edge contractions of G .

DEFINITION 9. A ΔY move takes a connected $(3,3)$ subgraph of graph G and replaces it with a connected $(4,3)$ graph. A $Y\Delta$ move does the reverse, taking a connected $(4,3)$ subgraph of G and replacing it with a connected $(3,3)$ graph.

DEFINITION 10. A graph G is planar if G can be represented in the plane with no intersections of edges except at vertices.

DEFINITION 11. A graph G is called complete if any two vertices in G are connected by an edge. A complete graph is denoted K_v , where v is the number of vertices in G .

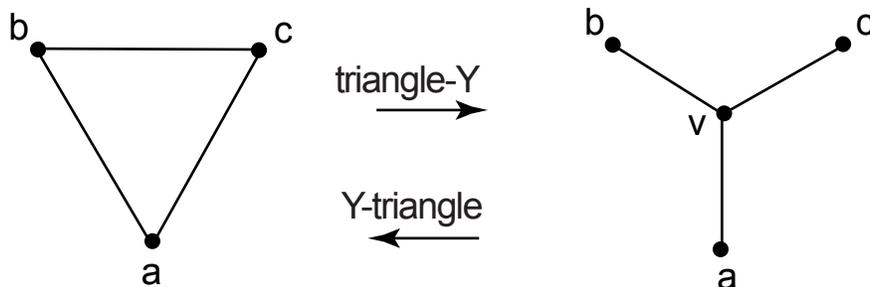


FIGURE 1. Pictured is an example of a $Y\triangle$ move and a $\triangle Y$ move (Figure from [GMN]).

DEFINITION 12. *The complement of a graph G on v vertices is the collection of vertices in G together with the edges not contained in G but contained in the complete graph K_v . We denote the complement of G as G^c .*

DEFINITION 13. *The degree of vertex v in G is the number of edges incident to v . We denote the minimum degree of G (i.e., the least degree among the vertices of G) as $\delta(G)$.*

DEFINITION 14. *A tree is a connected graph with no cycles.*

DEFINITION 15. *A spanning tree T of a graph G is a subgraph that is a tree containing all the vertices of G .*

DEFINITION 16. *A graph G is intrinsically knotted (IK) if every embedding of G contains a non-trivial knotted cycle.*

DEFINITION 17. *A graph G is minor minimal intrinsically knotted (MMIK) if it is IK and has no proper IK minor.*

LEMMA 18. $\triangle Y$ preserves IK.

PROOF. The proof is a consequence of Sachs' work [S]. □

Remark: The converse of this lemma is not true, as was first observed by Flapan and Naimi [FN].

DEFINITION 19. *A graph G is 2-apex if removing two vertices in G results in a planar graph.*

DEFINITION 20. A child G' of a graph G is a graph obtained by performing a ΔY move on G .

DEFINITION 21. The Euler characteristic of a graph G is the difference between the number of vertices and edges in G : $\chi(G) = v - e$.

PROPOSITION 22. [CMOPRW]

If G is (v, e) with $v > 6$ and $e > 5v - 15$, then G has a K_7 minor.

Remark: For a graph on nine vertices, we get the bound $e > 5 \times 9 - 15 = 30$. Hence, a $(9, e)$ graph where $e > 30$ has a proper IK minor (K_7) and is therefore not MMIK (note that it is IK, however). Consequently, we need only consider graphs with 30 or fewer edges.

THEOREM 23. [BBFFHL, OT]

If a graph G is 2-apex, then G is not IK.

LEMMA 24. If a $(9, 28)$ graph G is MMIK, then $\delta(G) \geq 3$. If a MMIK graph G is $(9, e)$ with $e = 29$ or $e = 30$, then $\delta(G) \geq 4$.

PROOF. Assume a $(9, e)$ graph G is MMIK, where $28 \leq e \leq 30$. If the minimum degree of G is zero, then there is an isolated vertex in G that can be deleted without affecting whether the graph is IK. If the minimum degree of G is one, then that vertex and its adjacent edge can also be deleted without affecting whether the graph is IK or not. Finally, if the minimum degree of G is two, then an edge contraction can be performed without affecting whether G is IK or not. In all three situations, we will have a minor G' of G such that G' is IK, which contradicts G being MMIK. Hence, $\delta(G) \geq 3$.

If G is a $(9, 30)$ graph or $(9, 29)$ graph and if G has a vertex of degree three, perform the $Y\Delta$ move on that vertex. This will give us a graph G' on eight vertices with at least 26 edges. The $Y\Delta$ move may introduce double edges, but no more than three. By Proposition 22, G' has a K_7 minor and is IK. Note that since we can obtain G from a ΔY move on G' , then G is a child of G' . Further, since G is MMIK, then, by [GMN, Lemma 1] G' is MMIK. However, we know the only MMIK graphs on eight vertices are H_8 and $K_{3,3,1,1}$ [BBFFHL, CMOPRW]. Since H_8 is an $(8, 21)$ graph and $K_{3,3,1,1}$ is an $(8, 22)$ graph, then G' cannot be either of these. Hence, G' is not MMIK, a contradiction. Thus, the minimum degree of G when G is a $(9, 29)$ graph or a $(9, 30)$ graph must be at least four. \square

LEMMA 25. If a connected graph G is a tree where G is not an isolated vertex, then $\delta(G) = 1$.

PROOF. Suppose $\delta(G) = 0$. Then, there is an isolated vertex in G , and since G is not simply an isolated vertex, G contains at least one more vertex. Consequently, G is not connected in this situation. Hence, $\delta(G) > 0$. Since G is connected, there is a path from any vertex in G to any other vertex in G . Consider the longest such path P_1 in G (i.e. the path containing the most vertices in G), with endpoints v_1 and v_2 . Suppose v_1 is of degree at least two. Then v_1 is attached to at least one vertex v_3 not in P_1 . Hence, there is a path P_2 from v_3 to v_2 consisting of P_1 , v_3 , and the edge between v_3 and v_1 . However, then P_2 is longer than P_1 (it contains one more vertex), a contradiction. Thus, v_1 is of degree one. Using a symmetric argument, v_2 is also of degree one. \square

LEMMA 26. *A connected graph G is a tree iff $\chi(G) = 1$.*

PROOF. Let n be the number of edges in a (v, n) graph G_n , where G_n is a tree and $n \leq 0$. If $n = 0$, then G_0 is an isolated vertex. Further, $\chi(G_0) = 1 - 0 = 1$. Now, let's assume that $\chi(G_k) = 1$ where G_k is a (v, k) graph and $0 \leq k \leq n$. Consider graph G_{k+1} with $k + 1$ edges. Note that every connected tree has at least one vertex of degree one (Lemma 25). Delete the edge adjacent to a vertex v' of degree one in G_{k+1} . Call the subgraph containing v' F . Now we have an isolated vertex v' and a subgraph H of G_{k+1} on k edges. Note that since G_{k+1} is a tree and contains no cycles, H will not contain any cycles and is consequently also a tree. Since H is a tree on k edges, $\chi(H) = 1$ by our induction hypothesis. Note also that $\chi(F) = 1$. Thus, since G_{k+1} is the collection of H , v' , and another edge, $\chi(G_{k+1}) = \chi(H) + \chi(F) - 1 = 1 + 1 - 1 = 1$. Hence, the induction hypothesis holds, and if any connected graph G is a tree, then $\chi(G) = 1$.

Now assume $\chi(G) = 1$ for some connected graph G . Assume there is a cycle in G containing vertex v_1 and another vertex v_2 . To create a cycle, there must be at least one more vertex v_3 in G . We now have a subgraph G' of G containing this cycle. The cycle in G may contain n vertices, v_1, v_2, \dots, v_n where $n \geq 3$. We begin at v_1 and travel along one edge to v_2 , then along another edge to v_3 , and so forth until we reach v_{n-1} . Clearly, we have traveled along $n - 1$ edges, and then we cross one more edge to return to v_n . Hence, there are n vertices and n edges in the cycle. Further, $\chi(G') = n - n = 0$. If we add a vertex to G' to begin reconstructing G , we must also add an edge, or our graph will not be connected. Hence, there is no way to construct G so that G contains a cycle and $\chi(G) = 1$. Hence, G must contain no cycles, and consequently, G is a tree. \square

LEMMA 27. *If a graph G is connected, there is a spanning tree T .*

PROOF. If G is a tree, then $G = T$.

Suppose G is not a tree. Then G contains at least one cycle C_1 . Remove any edge in C_1 to obtain a subgraph G_1 of G . If G_1 has a cycle C_2 , remove any edge in C_2 to obtain a subgraph G_2 of G . As $E(G)$ is finite, this process will terminate with a graph G_n that has no cycles. Then G_n is a tree and contains all the vertices in G . Further, $G_n = T$. \square

LEMMA 28. *If a graph G is connected, then $\chi(G) \leq 1$.*

PROOF. Suppose G is an (v, e) graph. Since G is connected, there is a spanning tree T containing all the vertices in G (due to Lemma 27). Due to Lemma 25, $\chi(T) = 1$. Hence, since G has no more vertices than T and no fewer edges than T , then $\chi(G) \leq \chi(T) = 1$. \square

PROPOSITION 29. *Graphs A_9 , B_9 , F_9 , and H_8 are MMIK.*

PROOF. Both H_8 and F_9 are obtained from K_7 by one or two ΔY moves. Since K_7 is IK ([CG]) and ΔY preserves IK Lemma 18, H_8 and F_9 are IK. The full proof that both are MMIK is given in [KS]. Similarly, A_9 and B_9 are descendants of the IK graph $K_{3,3,1,1}$. Due to work by [KS] and [F], both are MMIK. \square

There is one MMIK (9,28). See Figure nn in appendix. It's the graph whose complement is the disjoint union of a heptagon and K_2 .

PROPOSITION 30. *The (9,28) graph 260920 is MMIK.*

The proof is given in [GMN].

3. Overview of Thesis

In Chapter 2, we will present our research and proof that there exists only one MMIK graph on nine vertices and 28 or more edges. This chapter is divided into three sections. The first classifies (9,30) graphs, the second categorizes (9,29) graphs, and finally the third classifies (9,28) graphs.

Due to Proposition 22, we need only consider graphs with fewer than 31 edges.

PROPOSITION 31. *There are no MMIK (9,30) graphs.*

Sketch: We begin by showing that most 30 edge graphs have A_9 (which is MMIK) as a proper minor. The remaining (9,30) graphs are either 2-apex (making them not IK) or have either B_9 or K_7 as a proper minor (both of which are MMIK).

PROPOSITION 32. *There are no MMIK (9,29) graphs.*

Sketch: We used a similar strategy when looking at graphs on 29 edges: we showed that most have A_9 as a proper minor, some are 2-apex, and all but one of the rest of the graphs have B_9 or K_7 as a proper minor. The last graph has an unknotted embedding.

PROPOSITION 33. *There is only one MMIK (9,28) graph.*

Sketch: we found that a large number of graphs have F_9 as a proper minor. One graph is MMIK. The remaining graphs are either 2-apex, have B_9 or K_7 as a proper minor, or are subgraphs of the unknotted (9,29) graph.

In Chapter 3, we will discuss possibilities for further research. We will also suggest a strategy for classifying graphs with less than 28 edges.

Classification of Graphs

1. (9,30) Graphs

In this section we prove.

PROPOSITION 31. *There are no MMIK (9,30) graphs.*

LEMMA 34. *Let G be a (9,30) graph and H_i a component of G^c . Then $-2 \leq \chi(H_i) \leq 1$. If $\chi(H_i) = -2$, then H_i is K_4 .*

PROOF. For a given number of vertices, a complete graph will minimize $\chi(H_i)$, because complete graphs contain all possible edges. First, note that a complete graph on one vertex (K_1) has zero edges, so $\chi(K_1) = 1 - 0 = 1$. Since K_2 consists of two vertices and a single edge, then $\chi(K_2) = 2 - 1 = 1$. Next, we consider K_3 : $\chi(K_3) = 3 - 3 = 0$. For K_4 , $\chi(K_4) = 4 - 6 = -2$. We have used all our edges in this case, so $\chi(H_i) \geq -2$. As there are only six edges in G^c , any component H_i on more than four vertices will have $\chi(H_i) = v - e > v - 6 > -2$. So, if a component H_i has $\chi(H_i) = -2$, that component is K_4 . Due to Lemma 28, since G is connected, $\chi(G) \leq 1$. \square

LEMMA 35. *If G is (9,30), then G^c has between three and six components.*

PROOF. Since G is (9,30), G^c has nine vertices and six edges. Recall that $\chi(H_i) \leq 1$ due to Lemma 28. If G^c contained only one component, then $\chi(G^c) \leq 1$, but we know $\chi(G^c) = 3$, so this is not possible. Similarly, if there were exactly two components in G^c , since $\chi(G^c)$ is the sum of $\chi(H_i)$ for the two components, then $\chi(G^c) \leq 2$, which is again a contradiction. So G^c must have at least three components. To find an upper bound, we consider the case where K_4 is a component in the graph. If K_4 is a component in G^c , then the five other components are trees, and $\chi(G^c) = -2 + 1 + 1 + 1 + 1 + 1 = 3$. Note that if K_4 is a component in G^c , since it uses all six edges, the other five components are all single vertices (trees). In this case, we have six components.

By Lemma 35, a component with $\chi = -2$ is K_4 , so let's consider how many components can each have $\chi = -1$. A connected graph with $\chi = -1$ must be a (4,5) or a (5,6) graph. We can

only have one such component, as any more would result in more than six edges. There is only one edge left in the former, so there will be one (2,1) component and three isolated vertices. Here, G^c has five components. If the component is a (5,6) graph, there are no edges left, so the other components are isolated vertices, of which there are four. Again, there are five components. Lastly, we must consider how many components can each have $\chi = 0$. A connected graph having $\chi = 0$ is a (3,3) graph, K_3 , a (4,4) graph, a (5,5) graph, or a (6,6) graph. We can have at most two of these, in which case we have used all of our edges, and both will be K_3 graphs. There will be three other components that are all isolated vertices. Here, $\chi(G^c) = 0 + 0 + 1 + 1 + 1 = 3$, and we have five components. If one component is a (4,4) graph, a (5,5) graph, or a (6,6) graph, then the other components must be trees. Since $\chi(G^c) = 3$, there must be exactly three more components in G^c , and $\chi(G^c) = 0 + 1 + 1 + 1 = 3$. All three possibilities will result in four components total.

Hence, six is the maximum number of components in G^c . \square

We will consider G^c , the graph complement. Below is a list of cases that will be useful for our proof of Proposition 31. There are five 2-apex graphs, 51 graphs have A_9 as a minor, one has K_7 as a minor, and two have B_9 as a minor.

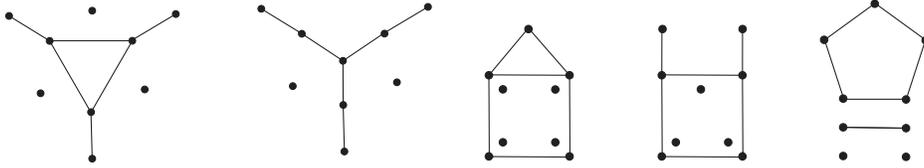


FIGURE 1. Pictured are the complement graphs of the five 2-apex (9,30) graphs.

1.1. Case 1: Three Components. There are 26 graphs in this case. We consider which graphs have A_9 as a proper minor, since A_9 is MMIK. We were able to show that the IK graphs that do not have A_9 as a proper minor have either B_9 or K_7 as a proper minor, both of which are MMIK. There are 25 such graphs that have A_9 as a proper minor. Another one is 2-apex. Below we considered subcases according to the number of edges in each of the three components.

Case 1.1: (0,0,6)

Since the first two components are single vertices, the third is a (7,6) graph. Eight of these have A_9 as a proper subgraph, and one is 2-apex.

Case 1.2: (0,1,5)

The first component is a single vertex and the second is K_2 , hence the third is a (6,5) graph. There are five of these graphs, and all have A_9 as a proper subgraph.

Case 1.3: (0,2,4)

The first component is an isolated vertex, the second is a tree with two edges, so the third is a (5,4) graph. There are three of these, and all have A_9 as a proper minor.

Case 1.4: (0,3,3)

The first component is a single vertex, which leaves eight vertices to split between the other two components. Since a connected graph with three edges can have at most four vertices, each of these components must have four vertices. There are only two connected (4,3) graphs, giving us three graphs for this case. All three have A_9 as a proper minor.

Case 1.5: (1,1,4)

The first two components are K_2 graphs. Consequently, the last component is a (5,4) graph. There are three of these, and all have A_9 as a proper minor.

Case 1.6: (1,2,3)

The first two components must contain five vertices together, so the third is a (4,3) graph. Hence, there are two such graphs. Both have A_9 as a proper minor.

Case 1.7: (2,2,2)

There is only one possible connected graph with only two edges (it has three vertices). Thus, there is only one graph in this category, and A_9 is a proper subgraph of it.

1.2. Case 2: Four Components. There are 25 graphs in this case. Of these graphs, 22 have A_9 as a proper subgraph. Three more are 2-apex.

Case 2.1: (0,0,0,6)

The first three components are single vertices, so the last is a (6,6) graph. There are 10 of these that have A_9 as a proper subgraph, and two more are 2-apex.

Case 2.2: (0,0,1,5)

The first three components contain together four vertices, so the fourth component is a (5,5) graph. One of these is 2-apex, and the other four have A_9 as a proper subgraph.

Case 2.3: (0,0,2,4)

The first three components contain five vertices together. Hence, the final component is a (4,4) graph. There are only two of these, and A_9 is a proper minor of both.

Case 2.4: (0,0,3,3)

The first two components are single vertices, so there are seven vertices to split between the last two components. Since a connected graph with three edges has either three or four vertices, one must be a (3,3) graph (which is a triangle), and the other is a (4,3) graph. There are two such graphs, and A_9 is a proper minor of both.

Case 2.5: (0,1,1,4)

There are cumulatively five vertices in the first three components, so the fourth component is a (4,4) graph. There are two graphs in this case, and A_9 is a proper minor of each.

Case 2.6: (0,1,2,3)

There are six vertices contained in the first three components, leaving three vertices for the last. Hence the last component is a triangle. There is only one such graph, and A_9 is a proper minor of this graph.

Case 2.7: (1,1,1,3)

The first three components contain six vertices together. Since the fourth component can only have three vertices, it is a triangle. There is only one graph in this case, and A_9 is a proper minor of it.

The following four component graphs would require more than nine vertices:

- (0,2,2,2)
- (1,1,2,2)

1.3. Case 3: Five Components. There are seven possible graphs in this category. Four of these graphs have A_9 as a proper minor. One graph has K_7 as a proper minor, one has B_9 as a proper minor, and one graph is 2-apex.

Case 3.1: (0,0,0,0,6)

The first four components are single vertices, so the fifth is a (5,6) graph. Two of these graphs have A_9 as a proper minor, one is 2-apex, one has B_9 as a proper minor, and one more has K_7 as a proper minor.

Case 3.2: (0,0,0,1,5)

The first three components are vertices of degree zero, the fourth is K_2 , and the last is a (4,5) graph. There is only one such graph, and A_9 is a proper minor of it.

Case 3.3: (0,0,0,3,3)

The first three components are single vertices, so there are six vertices left to split among the last two components. Since there are no connected graphs with three edges and less than three

vertices, both of the last two components must be triangles. Hence, there is only one graph in this case, and A_9 is a proper subgraph of it.

The following five component graphs would require more than nine vertices:

- (0,0,0,2,4)
- (0,0,1,1,4)
- (0,0,1,2,3)
- (0,0,2,2,2)
- (0,1,1,1,3)
- (0,1,1,2,2)
- (1,1,1,1,2)

1.4. Case 4: Six Components. There is only one graph with six components, and B_9 is its minor.

Case 4.1: (0,0,0,0,0,6)

There must be four vertices in the sixth component, making it a (4,6) graph. This component is K_4 . This graph has B_9 as a proper minor.

PROOF. (Proposition 31) Assume G is MMIK. G must have minimum degree of at least four according to Lemma 24. Since G is a (9,30) graph, it must be one of the graphs described in the above cases. However, each possible graph is either 2-apex or has an MMIK proper minor. Hence, G is not MMIK. \square

2. (9,29) Graphs

In this section we prove:

PROPOSITION 32. *There are no MMIK (9,29) graphs.*

LEMMA 36. *A component H_i of G^c has $-2 \leq \chi(H_i) \leq 1$.*

PROOF. First, note $\chi(G^c) = v - e = 9 - 7 = 2$. From Lemma 28, $\chi(H_i) \leq 1$. Now we will try to minimize $\chi(H_i)$. The maximum number of edges (and smallest $\chi(H_i)$) is $\frac{v(v-1)}{2}$ for the complete graph K_v , which has $\chi(K_v) = v - \frac{v(v-1)}{2}$. K_1 and K_2 are trees and $\chi(K_1) = \chi(K_2) = 1$. Next, $\chi(K_3) = 3 - 3 = 0$. A complete graph on four vertices has six edges (K_4), so $\chi(K_4) = 4 - 6 = -2$. For K_5 , $\chi(K_5) = 5 - \frac{5(4)}{2} = -5$. However, as G^c has seven edges, K_5 cannot be a component in G^c as it has $\frac{5(4)}{2} = 10$ edges. Hence, $-2 \leq \chi(H_i) \leq 1$. \square

LEMMA 37. G^c has between two and five components.

PROOF. Note $\chi(H_i) \leq 1$ (using Lemma 28). Since $\chi(G^c) = 2$ and $\chi(G^c)$ is the sum of $\chi(H_i)$ over all the components, then G^c must have at least two components. In order to maximize the number of components, we consider the case where one component has $\chi = -2$. Then there will be five components, and $\chi(G^c) = -2 + 1 + 1 + 1 + 1 = 2$. Note that the other components must be trees. We cannot have another non-tree component, as we have only one edge (in the case of K_4) and no edges (in the case of the (5,7) graph) to spread between the other components. A component with zero edges is a single vertex, which is a tree. A component with one edge is K_2 , which is also a tree. Hence, if one component H_i has $\chi(H_i) = -2$, there are five components in all. Next, if two distinct components H_i and H_j each are such that $\chi(H_i) = \chi(H_j) = -1$, then in order to not exceed nine vertices, they must both be (4,5) graphs, or one is a (4,5) graph and the other is a (5,6) graph (since no connected graph on three or fewer edges has $\chi = -1$). However, in both cases, we have used more than seven edges. Hence, we can have at most one component H_i with $\chi(H_i) = -1$. There will be two or fewer edges left. If the two edges are spread over two components, we will have two K_2 graphs, and the final component will be an isolated vertex. Then $\chi(G^c) = -1 + 1 + 1 + 1 = 2$, giving us four components. If H_i has two edges, we have a (3,2) graph which makes $\chi(H_i) = 3 - 2 = 1$. The other components are isolated vertices, and $\chi(G^c) = -1 + 1 + 1 + 1 = 2$, and again there are four components. Lastly, we consider how many components H_i with $\chi(H_i) = 0$ are possible. The graph with the fewest vertices having $\chi = 0$ is a (3,3) graph. There can either be two (3,3) graphs or one (3,3) graph and one (4,4) graph since G^c has seven edges. The other components cannot have $\chi = -2$, $\chi = -1$, or $\chi = 0$, so they must be trees. Again, we have four components: $\chi(G^c) = 0 + 0 + 1 + 1 = 2$. Hence, there are no more than five components in G^c . Moreover, G^c must have between two and five components. \square

Again, we consider G^c , which is a (9,7) graph. Listed below are cases that will be used for our proof of Proposition 32. There are 97 graphs having A_9 as a proper minor, 25 2-apex graphs, five graphs having B_9 as a proper minor, five graphs having K_7 as a proper minor, and one graph with an unknotted embedding.

2.1. Case 1: Two Components. There are 38 possible graphs in this case. Of these, 34 have A_9 as a proper minor, three are 2-apex, and one has K_7 as a proper minor.

Case 1.1: (0,7)

If there are zero edges in the first component, the component must be a tree consisting of a single vertex. This means the other component is an (8,7) graph. There are 15 such graphs that have A_9 as a subgraph. Two more are 2-apex, and another has K_7 as a proper minor.

Case 1.2: (1,6)

A component having one edge will be a (2,1) graph, of which there is only one (an edge with a vertex on each end). Hence, the second component must be a (7,6) graph. There are eight such graphs that have A_9 as a proper subgraph. One more is 2-apex.

Case 1.3: (2,5)

Again, there is only one connected graph consisting of two edges: two edges connected by a vertex, and each has a vertex on the disconnected end. This implies the second component is a (6,5) graph. There are five of these that have A_9 as a proper subgraph.

Case 1.4: (3,4)

A connected graph with three edges can either have three vertices (a triangle) or four vertices. In the former, this would mean the second piece is a connected (6,4) graph. There are no such graphs (a connected graph on v vertices has at least $v - 1$ edges according to Lemma 28), so we assume the piece with three edges has four vertices. There are two connected (4,3) graphs. The second component is a (5,4) graph, and there are three of these. In total, there are six graphs in this case, and all have A_9 as a proper subgraph.

2.2. Case 2: Three Components. There are 63 graphs in this case. Of these graphs, 49 have A_9 as a proper minor, and 12 are 2-apex. One has B_9 as a proper minor, and one more has an unknotted embedding.

Case 2.1: (0,0,7)

The first two components are single vertices, so the third is a (7,7) graph. There are 22 of these that have A_9 as a proper subgraph. Seven more are 2-apex graphs.

Case 2.2: (0,1,6)

There is one vertex in the first component and two in the second, thus the third is a (6,6) graph. There are eight such graphs that have an A_9 proper subgraph. Three others are 2-apex. The final graph, 260910, has an unknotted embedding (see figure below).

Case 2.3: (0,2,5)

The first two components contain four vertices together, so the third is a (5,5) graph. There are four of these that have an A_9 proper subgraph. One more is 2-apex.

Case 2.4: (0,3,4)

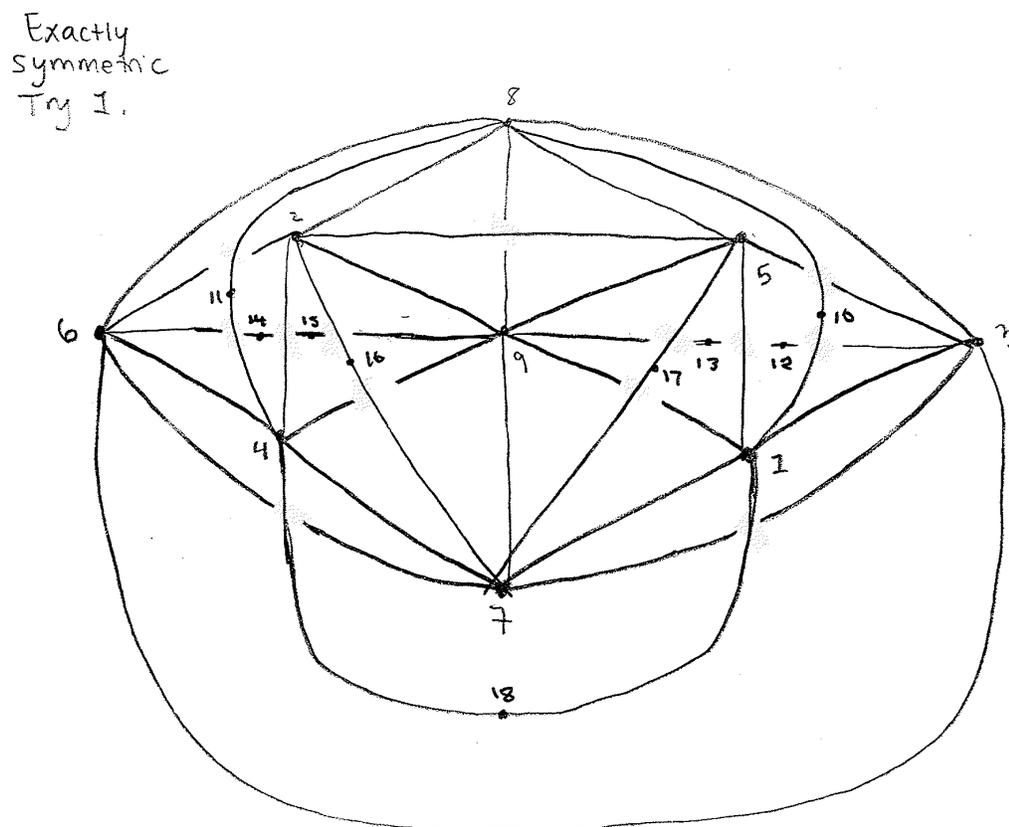


FIGURE 2. Above is a knotless embedding of the $(9,29)$ graph 260910.

The first component is a single vertex. The second is either a triangle or it has four vertices. In the first possibility, the third component would then be a $(5,4)$ graph, of which there are three. In the second, there are two $(4,3)$ graphs. The third component is a $(4,4)$ graph, of which there are two. In total, there are seven graphs in this case, all of which have an A_9 proper subgraph.

Case 2.5: $(1,1,5)$

The first two components have two vertices each, so the third is a $(5,5)$ graph. There are four of these with A_9 as a proper subgraph, and one is 2-apex.

Case 2.6: $(1,2,4)$

There are two vertices in the first component and three in the second, so the third is a (4,4) graph. There is one of these that has A_9 as a proper subgraph, and one more has B_9 as a proper subgraph.

Case 2.7: (1,3,3)

The first component contains two vertices, which leaves seven vertices for the other two components. Hence, one must be a triangle, and one must be a (4,3) graph. There are two such graphs, and both have A_9 as a proper subgraph.

Case 2.8: (2,2,3)

The first two components contain three vertices each, so the last must have three vertices. This means it is a triangle. Hence, there is only one such graph, and it has B_9 as a proper subgraph.

2.3. Case 3: Four Components. This case contains 27 graphs. There are 14 of these graphs that have A_9 as a proper minor, three have K_7 as a proper minor, two have B_9 as a proper minor, and eight are 2-apex.

Case 3.1: (0,0,0,7)

The first three components are single vertices, so the last is a (6,7) graph. There are eight of these having A_9 as a proper subgraph, six more are 2-apex, one has B_9 as a proper subgraph, and lastly, two have K_7 as a proper minor.

Case 3.2: (0,0,1,6)

The first three components contain four vertices together, so the last is a (5,6) graph. There are two such graphs that have A_9 as a proper subgraph. One more is 2-apex, another has B_9 as a proper subgraph, and the last has K_7 as a proper minor.

Case 3.3: (0,0,2,5)

The first three components have five vertices combined, so the fourth is a (4,5) graph. There is only one of these, and it has A_9 as a proper subgraph.

Case 3.4: (0,0,3,4)

The first two components contain two vertices together. The third may have either three or four vertices. If it has four, the last component must have three vertices and four edges. This is not possible, so the third component must be a triangle. This means the last component is a (4,4) graph. There are two of these, and both have A_9 as a proper subgraph.

Case 3.5: (0,1,1,5)

The first three components have together five vertices. The fourth then is a (4,5) graph. There is one such graph, and it has A_9 as a proper subgraph.

Case 3.6: (0,1,3,3)

The first two components have three vertices cumulatively, so there are six vertices in the last two components. Since there are no (2,3) graphs, both of these must be triangles. Consequently, there is one graph in this case, and it has A_9 as a proper subgraph.

The following four component graphs would require more than nine vertices:

- (0,1,2,4)
- (0,2,2,3)
- (1,1,1,4)
- (1,1,2,3)
- (1,2,2,2)

2.4. Case 4: Five Components. There are five graphs in this category. One has K_7 as a proper minor, two have B_9 as a proper minor, and two more are 2-apex.

Case 4.1: (0,0,0,0,7)

There are four single vertices and one component is a (5,7) graph. There is one graph that has a B_9 proper subgraph, two that are 2-apex, and the last has K_7 as a proper minor.

Case 4.2: (0,0,0,1,6)

There are cumulatively five vertices in the first four components, making the fifth component a (4,6) graph. There is only one such graph, and it has B_9 as a proper subgraph.

PROOF. (Proposition 32) Assume G is MMIK. G must have minimum degree four according to Lemma 24. Since G is a (9,29) graph, it must be one of the graphs described in the above cases. However, each possible graph is either 2-apex, has an MMIK proper minor, or is 260910 and has an unknotted embedding. Hence, G is not MMIK. \square

3. (9,28) Graphs

PROPOSITION 38. *There is only one MMIK (9,28) graph.*

LEMMA 39. *For a component H_i in G^c , $-3 \leq \chi(H_i) \leq 1$.*

PROOF. Recall that $\chi(H_i) \leq 1$ (Lemma 28). A complete graph maximizes the number of edges in a graph for a given number of vertices. Hence, we will consider Euler characteristic of complete graphs. K_3 has $\chi(K_3) = 3 - 3 = 0$, and K_4 has $\chi(K_4) = 4 - 6 = -2$. Since K_5 uses 10 edges, G^c does not have enough edges to have K_5 as a component. However, H_i could be a (5,8) graph, in which case $\chi(H_i) = 5 - 8 = -3$.

Hence, -3 is the minimum value for $\chi(H_i)$. \square

LEMMA 40. G^c has between one and five components.

PROOF. First note that $\chi(G^c) = 9 - 8 = 1$, and that $\chi(H_i) \geq -3$ (due to Lemma 39). Note $\chi(H_i) = -3$ if H_i is a (5,8) graph. We will consider the case when a (5,8) graph is a component in G^c as this will maximize the number of components in G^c . In this case, since all the edges of G^c are used in that single (5,8) component, the other components are isolated vertices, which are K_1 and $\chi(K_1) = 1$. Hence, $\chi(G^c) = -3 + 1 + 1 + 1 + 1 = 1$. This gives us five components: one (5,8) graph and four isolated vertices.

Suppose $\chi(H_i) = -2$. Then H_i is a (4,6) graph, a (5,7) graph, or a (6,8) graph. In the first case, there are two edges left. If both are in one component, it is a (3,2) graph, and there are two isolated vertices as the final components. This gives us a total of four components. If the two edges are spread over two components, those components are (2,1) graphs, and the final component is an isolated vertex. Again, there are four components total. If H_i is a (5,7) graph, there is one edge left. Then the other components are one (2,1) graph and two isolated vertices. Again, there are four components. Finally, if H_i is a (6,8) graph, there are no other edges left for the other components. Hence, there are three isolated vertices, giving us four components in total.

Now consider the case when $\chi(H_i) = -1$. Then H_i could be a (4,5), (5,6), (6,7), or (7,8) graph. There can only be one such component since G^c contains only eight edges. The connected graph with the fewest edges having $\chi(H_i) = 0$ is a (3,3) graph. In order to not exceed eight edges, there can be one component with $\chi = -1$ and one component with $\chi = 0$. In this case, one is a (4,5) graph and the other is a (3,3) graph. All edges are used here, so the final components are two isolated vertices. There are four components total then. If there is one component with $\chi(H_i) = -1$ and K_3 is not another component, then the remaining components must be trees. Since $\chi(G^c) = 1$, there must be exactly two more components, which makes a total of three components, and $\chi(G^c) = -1 + 1 + 1 = 1$.

Let's suppose that $\chi(H_i) = 0$. Then H_i is a (3,3), (4,4), (5,5), (6,6), (7,7), or an (8,8) graph. There can be at most two such graphs since G^c has eight edges. Consequently, the remaining components are all trees. Then either $\chi(G^c) = 0 + 0 + 1 = 1$ or $\chi(G^c) = 0 + 1 = 1$. There will be two or three components if $\chi(H_i) = 0$.

Thus, the maximum number of components in G^c is five. \square

LEMMA 41. If a MMIK (9,28) graph has minimum degree three, then it must contain K_4 .

PROOF. If G is a (9,28) graph and if G has a vertex of degree three, perform the $Y\Delta$ move on that vertex. This will give us a graph G' on eight vertices with at least 25 edges. The $Y\Delta$ move may introduce double edges, but no more than three. If there are exactly three double edges, G contains K_4 . If there are two or fewer double edges, then G' has at least 26 edges. By Proposition 22, G' has a K_7 minor and is IK. Note that since we can obtain G from a ΔY move on G' , then G is a child of G' . Further, since G is MMIK, then, by [GMN, Lemma 1] G' is MMIK. However, we know the only MMIK graphs on eight vertices are H_8 and $K_{3,3,1,1}$ [BBFFHL, CMOPRW]. Since H_8 is an (8,21) graph and $K_{3,3,1,1}$ is an (8,22) graph, then G' cannot be either of these. Hence, G' is not MMIK, a contradiction. Thus, if the minimum degree of G is three, then G must contain K_4 . \square

Below is a list of cases that we will use to prove Proposition 38. We consider G^c , a (9,8) graph. There are 168 graphs with F_9 as a proper minor, 111 2-apex graphs, four graphs with B_9 as a proper minor, eight graphs with K_7 as a proper minor, two graphs with an unknotted embedding, and the final graph is MMIK.

3.1. One Component. There are 35 graphs with minimum degree more than three in this case. There are 25 of these that have F_9 as a proper minor and the rest are 2-apex. Seven graphs have minimum degree equal to three.

Case 1.1: (8)

Since there is only one component, it must have eight edges, making it a (9,8) graph. There are 35 of these with minimum degree greater than three, 25 of which have F_9 as a proper minor and 10 which are 2-apex.

Seven more graphs have minimum degree three. One of these is 2-apex, two have F_9 as a proper minor, and the final four graphs do not contain K_4 .

3.2. Two Components. There are 135 graphs with two components and minimum degree more than three. Of these graphs, 89 have F_9 as a proper minor, and 43 are 2-apex. Two more have an unknotted embedding. Five graphs have minimum degree of three. The final graph is MMIK.

Case 2.1: (0,8)

These graphs consist of one vertex of degree zero and an (8,8) graph. Of these, 73 have minimum degree more than three. There are 45 graphs in this case with F_9 as a proper minor and 28 that are 2-apex.

There are 11 graphs that have minimum degree equal to three. Of these, three are 2-apex, seven have F_9 as a proper subgraph, and the final one does not contain K_4 .

Case 2.2: (1,7)

A component with one edge must have exactly two vertices. The second component then is a (7,7) graph. There are 29 of these, of which 17 have F_9 as a proper minor, 10 are 2-apex, one has 260910 as a super, and the final one, 260920, is MMIK.

Three graphs have minimum degree equal to three, and all have F_9 as a proper subgraph.

Case 2.3: (2,6)

A component with two edges has three vertices, making the second component a (6,6) graph. There are 12 such graphs. Eight of these have F_9 as a proper minor, three are 2-apex, and one more has 260910 as a super.

Only one graph has minimum degree of three and it has F_9 as a proper minor.

Case 2.4: (3,5)

A component with three edges is either a triangle or a (4,3) graph. Hence, the graphs in this case consist of either a (3,3) graph and a (6,5) graph or a (4,3) graph and a (5,5) graph. There are 15 of these graphs in total, of which 13 have F_9 as a proper minor and two are 2-apex.

One more graph has minimum degree three, and it has F_9 as a proper minor.

Case 2.5: (4,4)

A connected graph with four edges must have at least four vertices. Hence, since both components have four edges, these graphs must consist of a (4,4) graph and a (5,4) graph. There are six of these, all of which have F_9 as a proper minor.

3.3. Three Components. There are 98 graphs in this case. Of these graphs, 51 of these have F_9 as a proper minor, 40 are 2-apex, and six more have K_7 as a proper minor.

Case 3.1: (0,0,8)

These graphs consist of two isolated vertices and a (7,8) graph. There are 57 of these and 25 have F_9 as a proper minor, two have K_7 as a proper minor, and 29 are 2-apex.

Nine graphs have minimum degree of three. Three of these are 2-apex, four have F_9 as a proper minor, and the final two graphs do not contain K_4 .

Case 3.2: (0,1,7)

These graphs contain one isolated vertex, an edge with two vertices, and a (6,7) graph. There are 17 of these, of which eight have F_9 as a proper minor, one has K_7 as a proper minor, and eight more are 2-apex.

Two graphs have minimum degree of three and both have F_9 as a proper subgraph.

Case 3.3: (0,2,6)

A component with two edges must have exactly three vertices. Hence, these graphs contain one isolated vertex, one (3,2) graph, and a (5,6) graph. There are five of these graphs. Three have F_9 as a proper minor, one has K_7 as a proper minor, and one is 2-apex.

Case 3.4: (0,3,5)

A component with three edges is either a triangle or a (4,3) graph, making the third component either a (5,5) graph or a (4,5) graph. There are seven graphs in this category. Six have F_9 as a proper minor and one more is 2-apex.

Case 3.5: (0,4,4)

Since a component with four edges must have at least four vertices, the graphs in this category consist of an isolated vertex and two (4,4) graphs. There are three of these graphs, and two have F_9 as a proper minor while the last has K_7 as a proper minor.

Case 3.6: (1,1,6)

The first two components are (2,1) graphs, which makes the last a (5,6) graph. There are five graphs in this category. Three have F_9 as a proper minor, one has K_7 as a proper minor, and one more is 2-apex.

Case 3.7: (1,2,5)

The first component is a (2,1) graph, the second is a (3,2) graph, and the third is a (4,5) graph. There is only one such graph and it has F_9 as a proper minor.

Case 3.8: (1,3,4)

The first component is a (2,1) graph. In order to have no more than nine vertices, the next two components must be a (3,3) graph (a triangle) and a (4,4) graph. There are two graphs in this category, both of which have F_9 as a proper subgraph.

Case 3.9: (2,3,3)

The first component is a (3,2) graph making the other two components triangles. There is only one graph in this case, and it has F_9 as a proper minor.

The following three component graphs require more than nine vertices:

- (2,2,4)

3.4. Four Components. There are 25 graphs in this case. Three have an F_9 proper subgraph, two have K_7 as a proper minor, four have B_9 as a proper minor, and 16 are 2-apex.

Case 4.1: (0,0,0,8)

These graphs consist of three isolated vertices and a (6,8) graph. There are 18 of these graphs. Two have F_9 as a proper minor, one has K_7 as a proper minor, one has B_9 as a proper minor, and 14 others are 2-apex.

Four have minimum degree of three. One of these is 2-apex, one has F_9 as a proper minor, one has B_9 as a proper minor, and the final graph does not contain K_4 .

Case 4.2: (0,0,1,7)

These graphs have two isolated vertices, one (2,1) graph, and one (5,7) graph. There are four of these. One has K_7 as a proper minor, another one has B_9 as a proper minor, and two are 2-apex.

Case 4.3: (0,0,2,6)

These graphs contain two isolated vertices, one (3,2) graph, and one (4,6) graph (which is K_4). There is only one such graph, and it has B_9 as a proper minor.

Case 4.4: (0,0,3,5)

Since a component with five edges must have at least four vertices, the graphs in this case have two isolated vertices, one (3,3) graph, and one (4,5) graph. There is only one such graph and it has F_9 as a proper minor.

Case 4.5: (0,1,1,6)

These graphs contain one isolated vertex, two (2,1) graphs, and one (4,6) graph (which is K_4). There is only one such graph, and it has B_9 as a proper minor.

The following four component graphs require more than nine vertices:

- (0,0,4,4)
- (0,1,2,5)
- (0,1,3,4)
- (0,2,2,4)
- (0,2,3,3)
- (1,1,1,5)
- (1,1,2,4)
- (1,1,3,3)
- (1,2,2,3)
- (2,2,2,2)

3.5. Five Components. There are only two graphs in this case, and both are 2-apex.

Case 5.1: (0,0,0,0,8)

These graphs have four isolated vertices and one $(5,8)$ graph. There are only two such graphs, and both are 2-apex.

The following five component graphs require more than nine vertices:

- $(0,0,0,1,7)$
- $(0,0,0,2,6)$
- $(0,0,0,3,5)$
- $(0,0,0,4,4)$
- $(0,0,1,1,6)$
- $(0,0,1,2,5)$
- $(0,0,1,3,4)$
- $(0,0,2,2,4)$
- $(0,0,2,3,3)$
- $(0,1,1,1,5)$
- $(0,1,1,2,4)$
- $(0,1,1,3,3)$
- $(0,1,2,2,3)$
- $(0,2,2,2,2)$
- $(1,1,2,2,2)$

PROOF. (Proposition 38) If G is a $(9,28)$ MMIK graph then G has minimum degree of at least three (according to Lemma 24). Further, it must be one of the graphs described above. If it is not 260920, then it either has a proper MMIK minor or has 260910 as a super, or is 2-apex. Hence, G must be 260920. \square

Further Research

For further research, we would first like to determine which graphs on nine vertices may result in an MMIK graph. Considering graphs on 20 or fewer edges, we know there are no MMIK graphs [M]. For graphs on 21 edges, it has already been shown that there are 14 MMIK graphs ([BM] and [LKLO]). Our work has now shown that there is only one MMIK graph on 28 edges, and there are no MMIK graphs on 29 or 30 edges. Finally, due to previous research, we know that any graph on nine vertices and more than 30 edges must have a proper IK minor [CMOPRW]. Hence, there are no such graphs that are MMIK. Next, we must consider graphs on nine vertices and between 22 and 27 edges. We already know of such MMIK graphs on 22 edges. We believe that there are no new MMIK graphs having between 22 and 27 edges due to results obtained through a computer program [MS]. However, this must be mathematically verified.

We do not only want to know the number of edges MMIK graphs on nine vertices can have, but we also are interested in finding all such MMIK graphs. The 14 MMIK graphs on 21 edges are known, and these include F_9 and H_9 . The two MMIK graphs on 22 edges are A_9 and B_9 . The (9,28) graph 260920 was previously shown to be MMIK [GMN]. If it is found that there may be new MMIK graphs on nine vertices, these graphs must be fully described.

Another question is whether there exists any MMIK graph on 10 vertices. Again, we know that any such graph on more than 35 edges will have K_7 as a proper minor [CMOPRW]. However, for graphs with 35 or fewer edges, the possibility of minor minimal intrinsic knottedness must still be explored.

Our work has presented a strategy for finding MMIK graphs. In general, graphs seem to fall into two classes: they are 2-apex with one of the (9,30) 2-apex graphs as a super, or they are IK and have A_9 , B_9 , F_9 , H_8 , or K_7 as a minor. When considering graphs with nine vertices and between 22 and 27 edges, 1377 of these graphs have H_8 as a proper minor [W]. A small number (45) of graphs have A_9 as a minor but not H_8 . A good strategy would be to determine first whether a graph in this category has H_8 as a minor. If not, we can check if a graph has A_9 as a minor. Then it can be determined whether the remaining graphs, of which there are 122, have B_9 , F_9 , or K_7 as a minor.

Most of the rest of the graphs will be 2-apex, and we can attempt to show this by showing these graphs are minors of one of the five 2-apex $(9,30)$ graphs. Any remaining cases can be dealt with individually.

Appendix: Drawings of Graph Complements

1. (9,30) Graphs

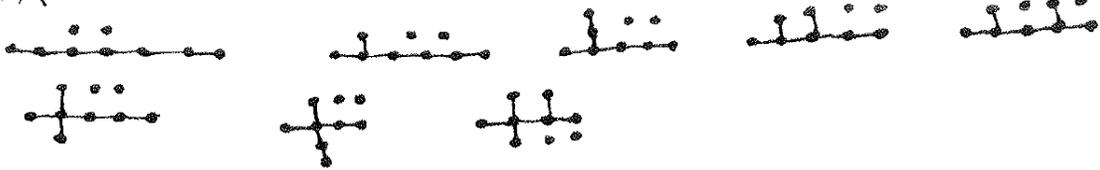
(9,30)

Case 1.1: (0,0,6)

2-apex

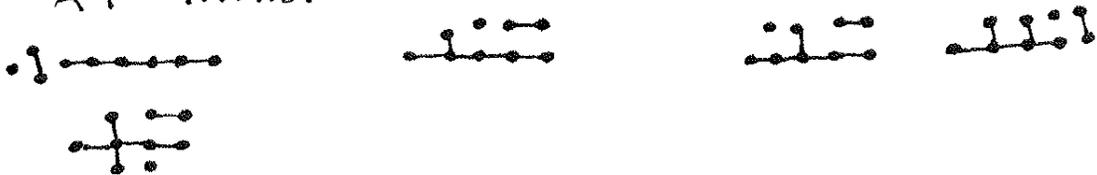


A9 minor



Case 1.2: (0,1,5)

A9 minor



Case 1.3: (0,2,4)

A₉ minor



Case 1.4: (0,3,3)

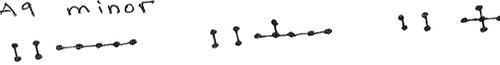
A₉ minor



(0,3,0)

Case 1.5: (1,1,4)

A₉ minor



Case 1.6: (1,2,3)

A₉ minor



Case 1.7: (2,2,2)

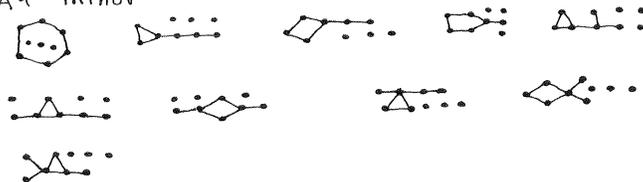


Case 2.1 (0,0,0,6)

2-apex



A₉ minor



(1,30)

Case 2.2: (0,0,1,5)

2-apex



A9 minor



Case 2.3: (0,0,2,4)

A9 minor



Case 2.4: (0,0,3,3)

A9 minor



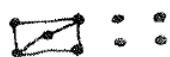
Case 2.5: (0,1,1,4)

A9 minor



$(9, 30)$ Case 2.6: $(0, 1, 2, 3)$ A₉ minorCase 2.7: $(1, 1, 1, 3)$ A₉ minorCase 3.1: $(0, 0, 0, 0, 6)$

2-apex

A₉ minorB₉ minorK₇ minor

$(9,30)$

Case 3.2: $(0,0,0,1,5)$

A_9 minor



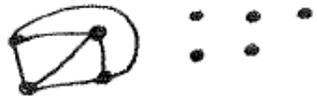
Case 3.3: $(0,0,0,3,3)$

A_9 minor



Case 4.1: $(0,0,0,0,0,6)$

B_9 minor



2. (9,29) Graphs

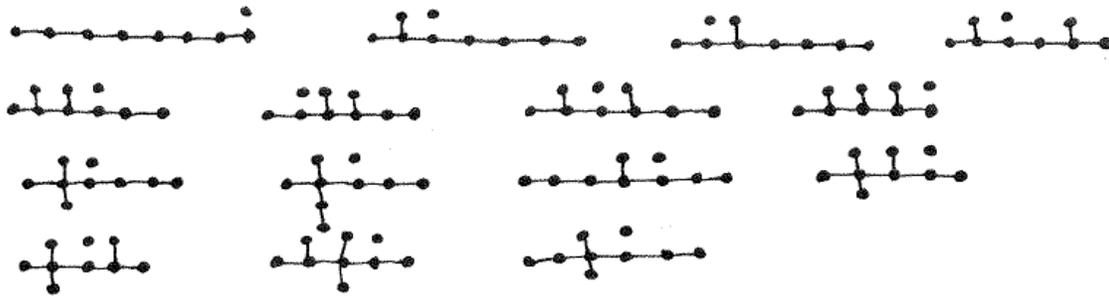
(9,29)

Case 1.1: (0,7)

2-apex



A_9 minor



K_7 minor



Case 1.2: (1,6)

2-apex



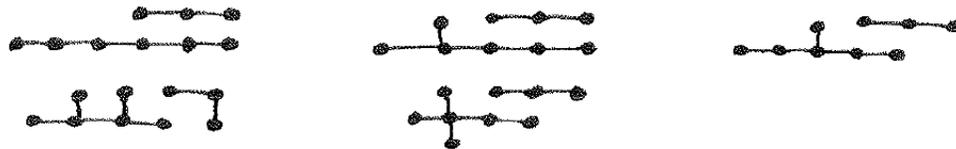
A9 minor



(9,29)

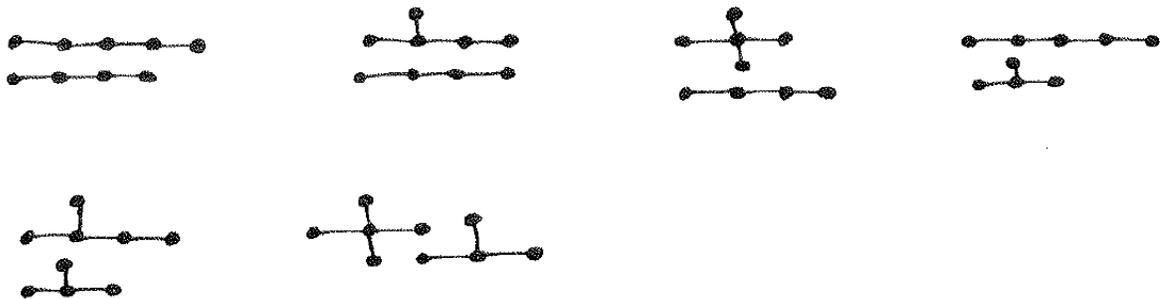
Case 1.3: (2,5)

A9 minor



Case 1.4: (3,4)

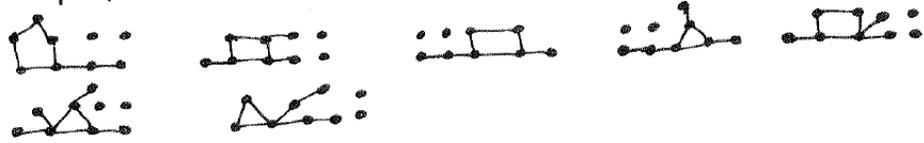
A9 minor



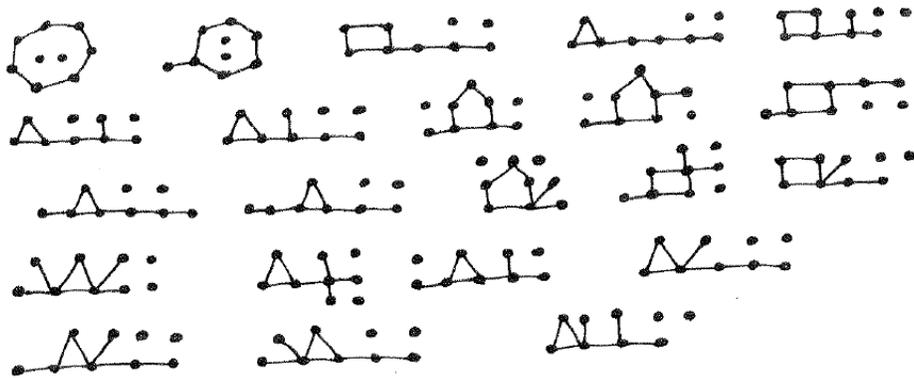
(9, 29)

Case 2.1: (0, 0, 7)

2-apex



A9 minor

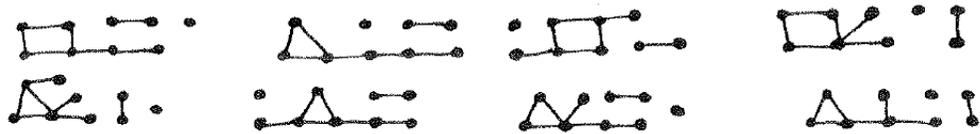


Case 2.2: (0, 1, b)

2-apex



A9 minor



unknotted embedding



(9,29)

Case 2.3: (0,2,5)

2-apex

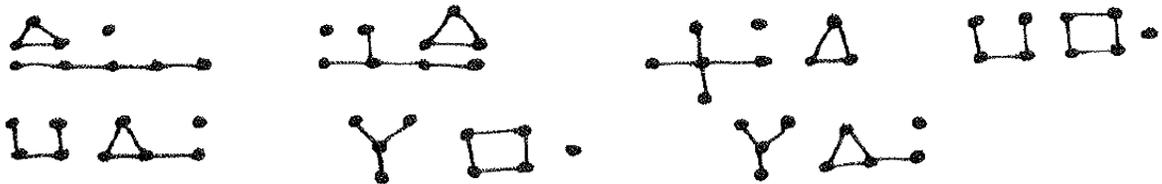


A9 minor



Case 2.4: (0,3,4)

A9 minor



Case 2.5: (1,1,5)

2-apex



A9 minor



Case 2.6: (1, 2, 4)

A₉ minor



B₉ minor



(9, 29)

Case 2.7: (1, 3, 3)

A₉ minor



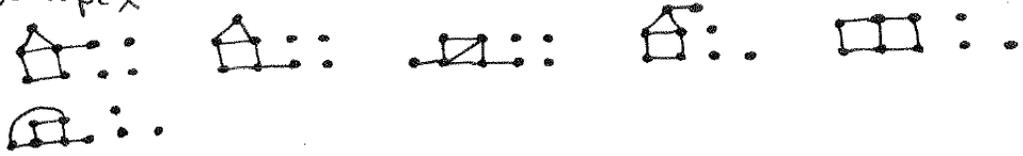
Case 2.8: (2, 2, 3)

B₉ minor

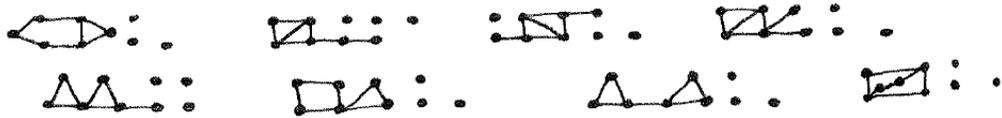


Case 3.1: (0,0,0,7)

2-apex



A9 minor



B9 minor



K7 minor



(9,29)

Case 3.2: (0,0,1,6)

2-apex



A9 minor



B9 minor



K7 minor



Case 3.3: $(0, 0, 2, 5)$
 A_9 minor



Case 3.4: $(0, 0, 3, 4)$

A_9 minor



Case 3.5: $(0, 1, 1, 5)$

A_9 minor



Case 3.6: $(0, 1, 3, 3)$

A_9 minor



(9, 29)

Case 4.1: $(0, 0, 0, 0, 7)$

2-apex



B_9 minor



K_7 minor



Case 4.2: $(0, 0, 0, 1, 6)$

B_9 minor



3. (9,28) Graphs

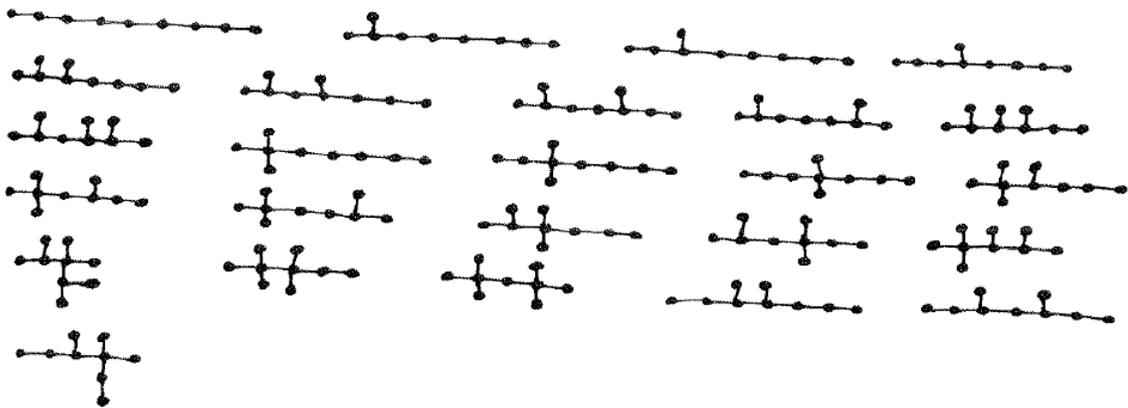
(9,28)

Case 1.1:(8)

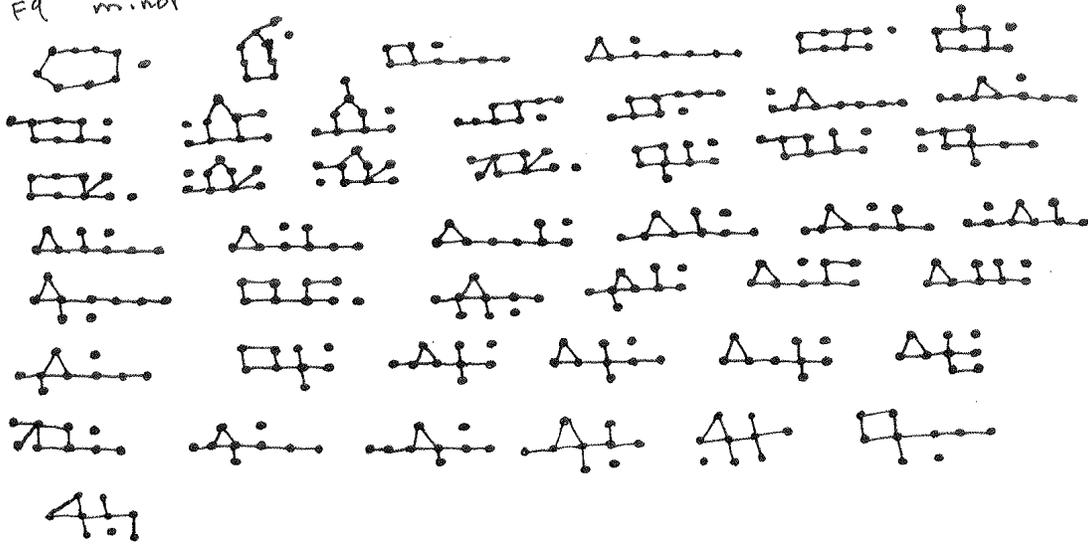
2-apex



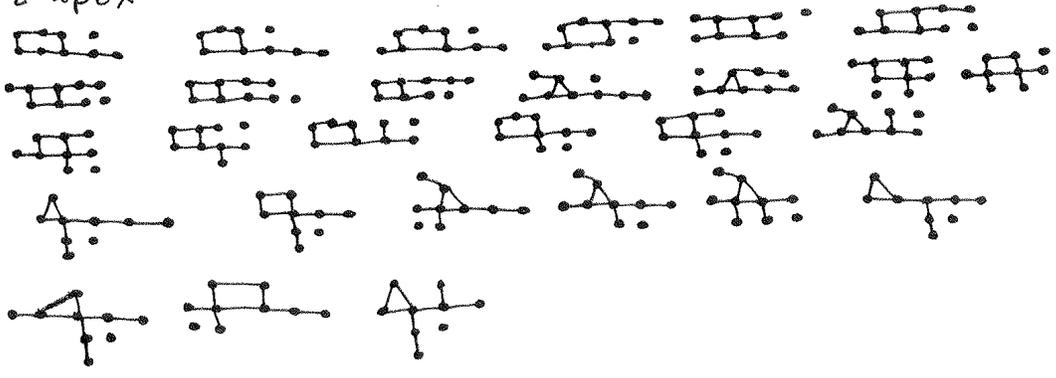
F9 minor



(9,28)
 Case 2.1: (0,8)
 F9 minor



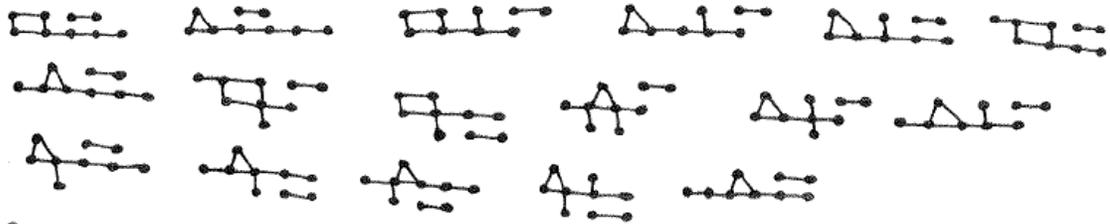
2-apex



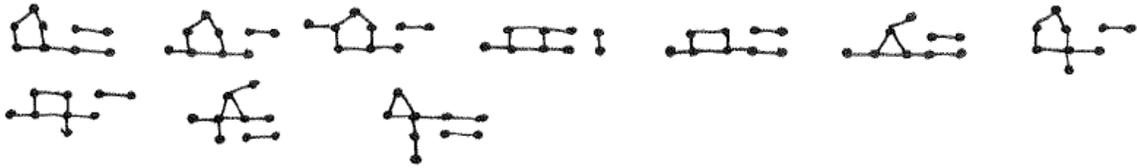
(9,28)

Case 2.2: (1,7)

F9 minor



2-apex



Unknotted Embedding



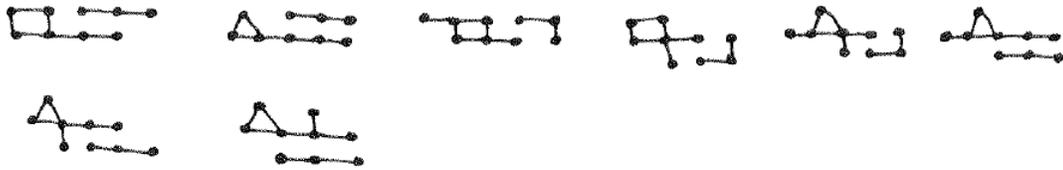
MMIK



(9,28)

2.3 : (2,6)

F9 minor



2-apex

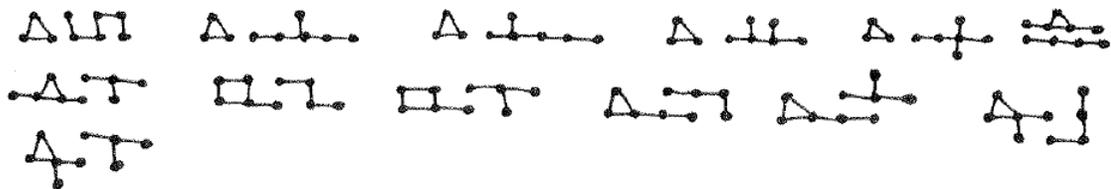


Unknotted Embedding



Case 2.4 : (3,5)

F9 minor



2-apex



(9,28)

Case 2.5 : (4,4)

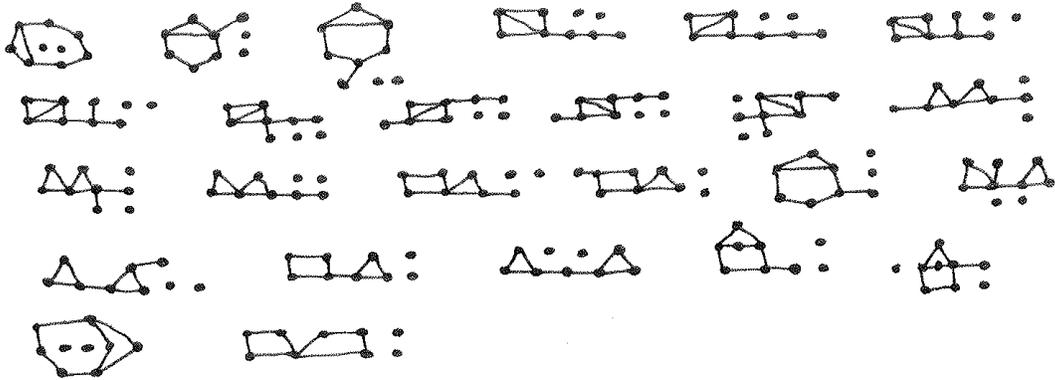
F9 minor



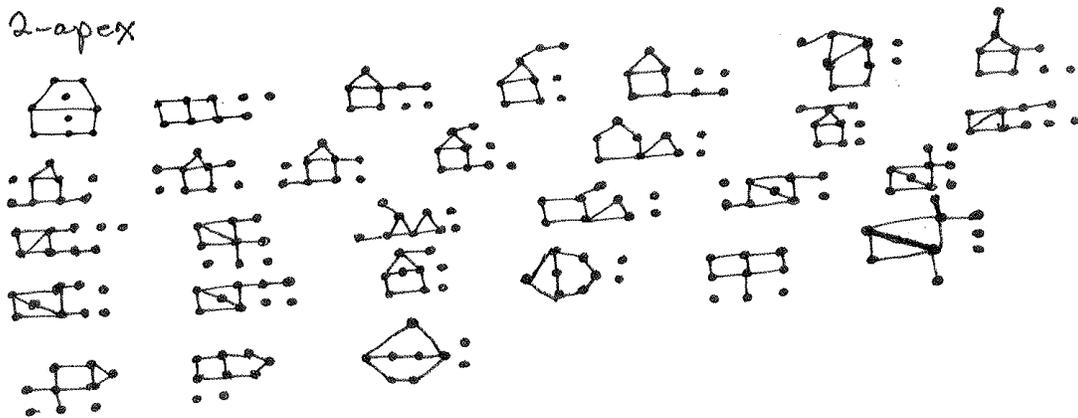
(9,28)

Case 3.1: (0,0,8)

F9 minor



2-apex



K7 minor



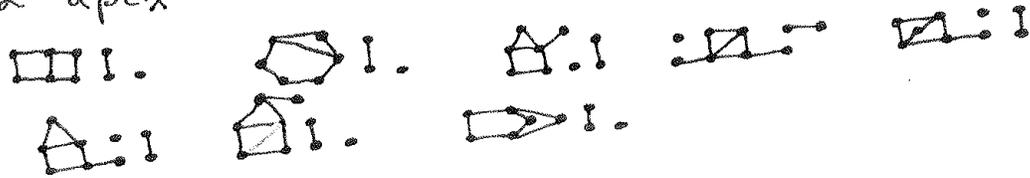
(9,28)

Case 3.2: (0,1,7)

F₉ minor



2-apex



K₇ minor



Case 3.3: (0,2,6)

F₉ minor



2-apex

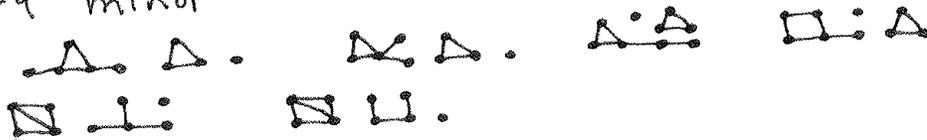


K₇ minor



Case 3.4: (0,3,5)

F₉ minor



2-apex



(9,28)

Case 3.5: (0,4,4)

F9 minor



K7 minor



Case 3.6: (1,1,6)

F9 minor



2-apex



K7 minor



Case 3.7: (1,2,5)

F9 minor



Case 3.8: (1,3,4)

F9 minor



Case 3.9: (2,3,3)

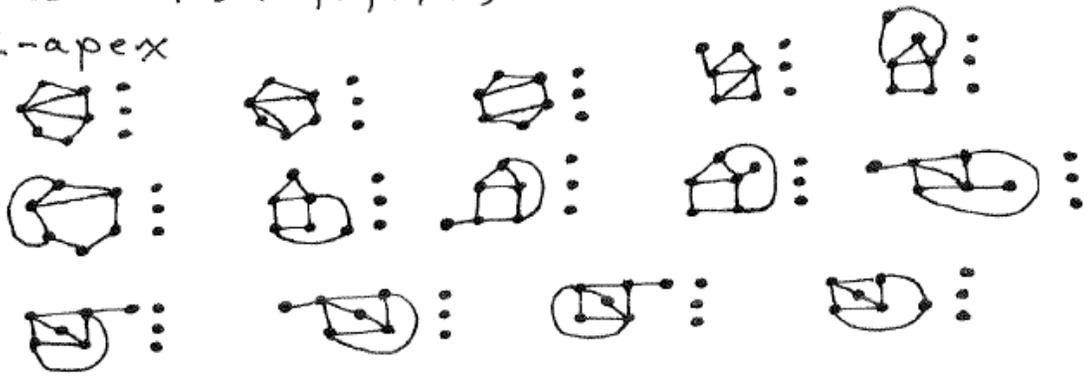
F9 minor



(9,28)

Case 4.1: (0,0,0,8)

2-apex



F_9 minor



B_9 minor



K_7 minor



(9,28)

Case 4.2: (0,0,1,7)

2-apex



B9 minor



K7 minor



Case 4.3: $(0, 0, 2, 6)$

B_4 minor



Case 4.4: $(0, 0, 3, 5)$

F_9 minor



Case 4.5: $(0, 1, 1, 6)$

B_4 minor



$(9, 28)$

Case 5.1: $(0, 0, 0, 0, 8)$

2-apex



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