Minor Minimal Intrinsically Knotted Graphs with 21 Edges

Jamison Barsotti

Abstract. In this paper, we prove that the only minor minimal intrinsically knotted graphs with 21 edges are the fourteen graphs that are obtained through a series of Triangle-Y moves from the complete graph on 7 vertices.

CHAPTER 1

Introduction

A graph G is **intrinsically knotted** (IK) if every tame embedding of G in R^3 contains a non-trivial knotted cycle. In 1983, Conway and Gordon showed that the complete graph on seven vertices, K_7, is IK [**CG**], as well as showing that deleting any edge from K_7 will provide a graph that is not IK. This implies that K_7 is not only IK but, since no proper minor of K_7 will be IK, K_7 is said to be **minor minimal IK** or MMIK. That there is a finite number of MMIK graphs follows from work of Robertson and Seymour [**RS**].

That an IK graph must have at least 21 edges was shown, independently, by Mattman [**M**] and Johnson, Kidwell, and Michael [**JKM**]. Since, K_7 has 21 edges, this paper analyzes the graphs with 21 edges and provides a complete list of MMIK graphs on 21 edges. In his paper, Mattman also analyzes graphs with 21 edges and 9 vertices or less, so this paper considers the graphs on 21 edges and more than 9 vertices.

When considering IK graphs, a method called the **Triangle-Y** (TY, or ΔY) move is used frequently to obtain new graphs that are also IK [**S**]. The TY move takes three mutually adjacent vertices and replaces their adjoining edges with a vertex whose neighborhood consists of the three formerly adjacent vertices. It is worth noting that if one instance of TY is applied to a graph, G, the graph obtained G' will have the same number of edges as G and one more vertex and TY does not introduce any vertices of degree zero. Hence if we apply TY to K_7 we not only obtain another IK graph, we obtain another MMIK graph since any proper minor of this new graph will be obtained by removing or contracting at least one edge and a graph must have at least 21 edges in order to be IK. We make this claim on any of the descendants - the graphs obtained through any number of TY moves - of K_7, since a vertex of degree zero cannot be introduced.

With the help of TY to establish a list of MMIK graphs, we will also consider graphs that become planar after the removal of two vertices. Such graphs are called 2**-apex**. If a graph is 2-apex, then it is not IK ([**BBFFHL**] and [**OT**]), hence we will show that many of the graphs with 21 edges are 2-apex, thus not IK.

Along the way, we also characterize some 1-apex - graphs that become planar after removing 1 vertex.

CHAPTER 2

Definitions and Lemmas

Throughout this paper, we will refer to a family of graphs called Heawood graphs. These are the 20 graphs obtained through a series of ΔY and $Y\Delta$ moves on the graph K_7. Fourteen of these can be obtained through only ΔY moves.

Consider a graph G. if a is in the set of vertices of G ($V(G)$) we denote the **degree** of a by the number of edges a is adjacent to, or equivalently for graphs without multiple edges, by the number of neighbors a has.

DEFINITION 1. *Recall the definition of Triangle-Y from the introduction. Conversely, when we perform the* **Y-Triangle**— *move, we are replacing a vertex whose degree is three with adjacencies between each pair of its neighbors.*

DEFINITION 2. *The* **Heawood graphs** *are the* 20 *graphs obtained through a series of* ΔY *and* $Y\Delta$ *moves on the graph* K_7. *Fourteen of these can be obtained through only* ΔY *moves.*

DEFINITION 3. *Consider graph* G. *A graph that is obtained through any number of vertex contractions, vertex deletions, or edge deletions, is called a* **minor** *of* G. *A* **proper minor** *of* G *is a minor of* G *that is not* G.

DEFINITION 4. *A graph,* G*, is* **planar** *when it can be drawn in the plane with no edge crossings except at its vertices. Equivalently,* G *is planar if and only if it does not have* K_5 *or* $K_{3,3}$ *as a minor. A graph is called* **nonplanar** *if it is not planar.*

Consider the graphs H and G. The graph where each vertex in G contains all of H in its neighborhood will be denoted by $\mathbf{H} * \mathbf{G}$

A **cycle** is a sequence of vertices that starts and ends with the same vertex, no vertices are repeated except the first and last, and, with the exception of the first vertex, each vertex is adjacent to the previous vertex in the sequence. Recall from the definition that a graph G is IK when any tame embedding in R^3 contains a non-trivial knotted cycle.

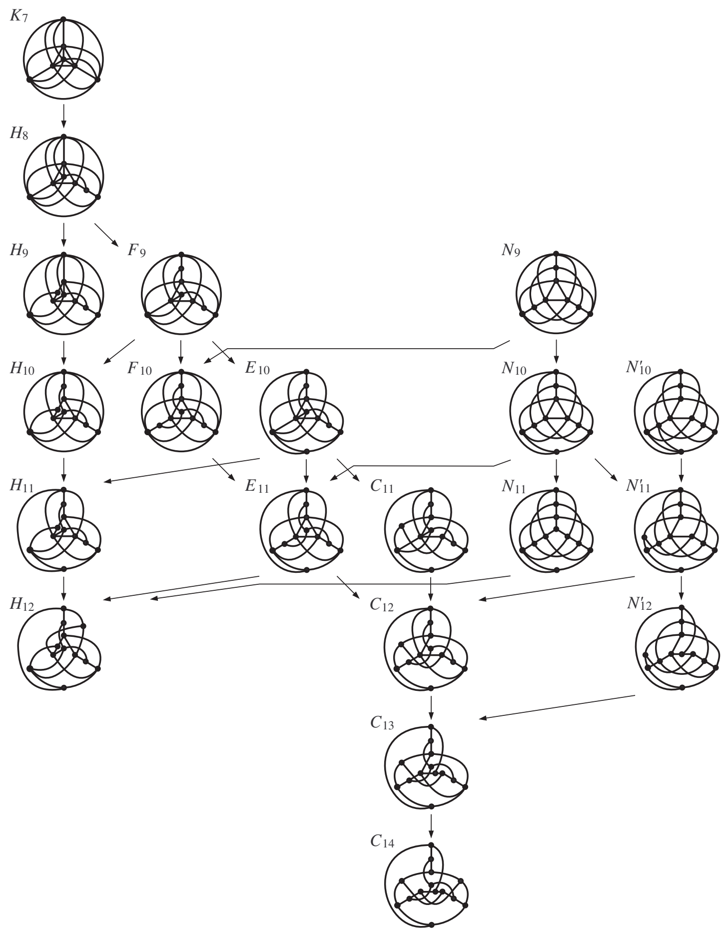

FIGURE 2.1. The Heawood graphs (figure taken from [**HNTY**]).

LEMMA 5. *A graph of the form* $H * K_2$ *is IK if and only if* H *is non-planar.*

REMARK 6. *This is due independently to* [**BBFFHL**] *and* [**OT**].

In the introduction, we talked of a graph being 2-apex when, if two vertices and their adjacent edges are removed, the resulting graph is planar.

LEMMA 7. *A* 2*-apex graph is not IK.*

PROOF. This is an immediate consequence of Lemma 5. □

LEMMA 8. *A graph on 20 or fewer edges is* 2*-apex.*

REMARK 9. *This is proved in* [**M**].

LEMMA 10. *A graph on* 20 *or fewer edges is not IK.*

PROOF. Combine Lemmas 8 and 7. □

DEFINITION 11. *A graph G is* **intrinsically linked** *(IL) when any tame embedding in R^3 contains a pair of nontrivially linked cycles.*

LEMMA 12. *The ΔY move preserves IK.*

REMARK 13. *Sach's* [**S**] *proved this for IL and the same argument shows it for IK.*

We say a graph is **minor minimal**, with respect to a certain property, if it has the property and none of its proper minors have the property.

DEFINITION 14. *A graph G is* **minor minimal IK** *(MMIK) if G is IK and none of the proper minors of G are IK.*

LEMMA 15. *There are 14 MMIK Heawood graphs, the descendants of K_7.*

PROOF. Since K_7 is IK and ΔY preserves IK then there are at least 14 Heawood graphs that are IK. Since each graph is connected, then any minor will require at least one edge to be deleted or contracted. Since any graph on 20 or fewer edges is not IK (Lemma 10), these 14 IK graphs are MMIK. The other 6 Heawood graphs can be shown to be not IK.

Figure 2.2 shows that N_{11} and $N\prime_{12}$ are not IK. Since the other four Heawood graphs obtain these graphs from a series of TY moves, they are also not IK. This was proved by Goldberg, Mattman, and Naimi [**GMN**] and [**HNTY**] independently.

□

Throughout this paper, we will use the convention $G - a$ and $G - a, b$ to denote the graph obtained by deleting vertex a and vertices a and b from the graph G, respectively. We will also write $G + a$ to denote the graph obtained when we add vertex a to G.

A **tree** is a connected graph where the deletion of any edge will result in a multicomponent graph. The **leaves** on a tree are exactly the vertices of degree one.

When we refer to a **triangle** on a graph we are talking about three mutually adjacent vertices in that graph.

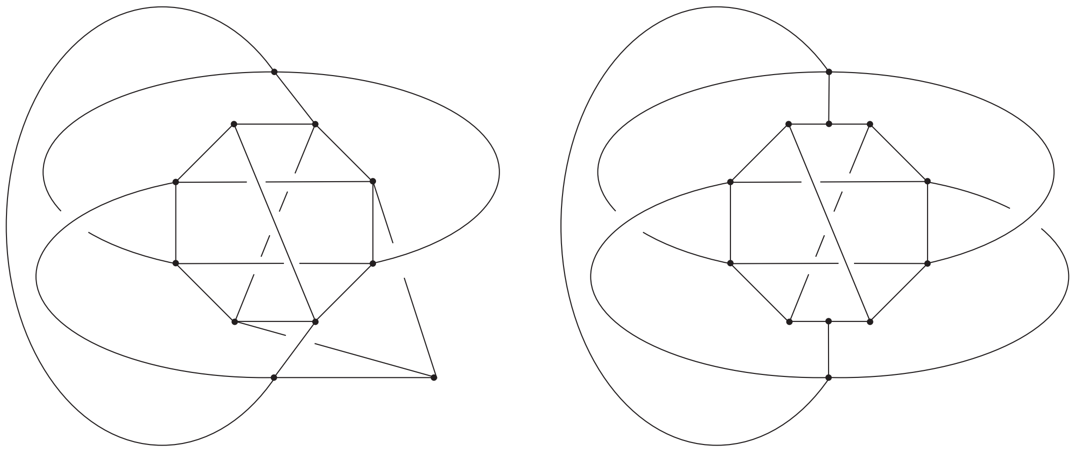

FIGURE 2.2. Embeddings of N_{11} and $N\prime_{12}$ that have no knotted cycles.[**GMN**]

LEMMA 16. *Let G be a graph with minimum degree 3 ($\delta(G) = 3$). If we can remove two vertices, a and b, such that $G - a, b = T \sqcup G'$ with T a tree of at most three vertices, then G has a triangle.*

PROOF. Let T be the tree of at most three vertices that is a component of $G - a, b$, where $\delta(G) = 3$. Since the minimum degree of G is 3, both a and b are adjacent to the leaves of T. In the case where T is of degree three, then at least a or b is adjacent to all three vertices of T. Hence, G has a triangle. □

A graph is called **complete** if every pair of vertices are mutually adjacent. The complete graph with n vertices is denoted K_n If G is a **bipartite** graph then G split into two parts where each vertex of G only has neighbors in the opposite part. The graph $\mathbf{K_{3,3}}$ is a bipartite graph where each part has three vertices and each vertex is adjacent to every vertex in the opposite part . $K_{3,3}$ and K_5 are the two minor minimal nonplanar graphs.

DEFINITION 17. *If we replace a vertex v with a pair of adjacent vertices that share the neighbors of v, we call that a* **vertex split***.*

DEFINITION 18. *Let v be a vertex of G, that has degree 2.* **Smoothing** *v replaces v and its adjacent edges with an edge adjacent to the neighbors of v. If such a move results in a multiedge we simply just remove v and its adjacent edges and don't replace it with anything.*

DEFINITION 19. *Let G be a graph. If H is obtained by the removal of all vertices of G that have degree less than three, by either deleting all degree zero vertex, deleting all degree one vertex and* *their adjacent edge, or smoothing all degree two vertices, we call H the* **topological simplification** *of G.*

DEFINITION 20. *Graphs G and H are* **topologically equivalent** *if there is an isomorphism between their topological simplifications.*

DEFINITION 21. *A* **split** *$K_{3,3}$ is a graph G formed from $K_{3,3}$ by a finite sequence of vertex splits. This means that G is topologically equivalent to $K_{3,3}$. In the process of topologically simplifying, we repeatedly delete vertices of degree 1 as well as those of degree 2 by smoothing. Then, in G, we can identify the six* **original vertices** *as those that are not deleted by topological simplification. An* **original** 4–**cycle** *is a cycle C in G that passes through exactly four original vertices. The* **split** 4–**cycle** *of C is the component of C in $G - v, w$ where v and w are the two original vertices not in C.*

LEMMA 22. *A graph G is a split $K_{3,3}$, if and only if, it is connected with a $K_{3,3}$ minor and $\chi(G) = -3$.*

PROOF. Assume G is a split $K_{3,3}$. Since G can be made using a series of vertex splits on a $K_{3,3}$ graph, then it is connected and has a $K_{3,3}$ minor. Since each vertex splits add exactly one vertex and one edge, $\chi(G) = \chi(K_{3,3}) = -3$.

Now assume G has a $K_{3,3}$ minor, is a single component, and that $\chi(G) = -3$. So G can be built by adding vertices and edges to a $K_{3,3}$. Since G is connected, for each added vertex, there is an added edge. Since $\chi(G) = \chi(K_{3,3}) = -3$, for each added edge, there is an added vertex. Hence, G is obtained from $K_{3,3}$ by a series of vertex splits. Thus, G is a split $K_{3,3}$.

□

DEFINITION 23. *A graph is* **1-apex** *if removing one vertex and its adjacent edges results in a planar graph.*

DEFINITION 24. *A* **path** *is a sequence of vertices where no vertices are repeated and each vertex in the sequence, except the first, is adjacent to the vertex prior to it.*

LEMMA 25. *Let G be a split $K_{3,3}$. The graph $G + a$ is 1–apex if there is an original vertex, v, such that any path from a to v contains another original vertex.*

From here on, we may denote the set of vertices of a given graph G, by $V(G)$.

Proof. Consider $G + a$ where G is a split $K_{3,3}$, and say that $v \in V(G)$ is an original vertex such that any path from a to v contains another original vertex. If we remove an original vertex w that is adjacent to v in $K_{3,3}$ then a is only adjacent to vertices on the split 4-cycle in $G - v, w$. Then $(G + a) - w$ is planar and $G + a$ is 1- apex. □

When discussing a planar graph G and its representation in the plane, we may reference the regions of G by talking about the areas bounded by edges of the graph, including the region that is not surrounded by edges but rather surrounds a boundary of edges. We will call this the **outer region** and the edges and vertices is surrounds will be called the **outer face**.

Remark 26. *Lemma 25 generalizes in an obvious way when G is the disjoint union of a split $K_{3,3}$ and a graph H such that $H + a$ has a planar diagram with a on the outer face.*

Lemma 27. *Suppose G is not 2–apex and $G^* = G - a, b$ is a split $K_{3,3}$ for some $a, b \in V(G)$. Then, in $G^* + a$, for each original vertex v, there is a path from a to v that avoids the other original vertices, and similarly for $G^* + b$.*

Proof. Since G is not 2–apex, $G^* + a$ is not 1-apex. Apply Lemma 25. □

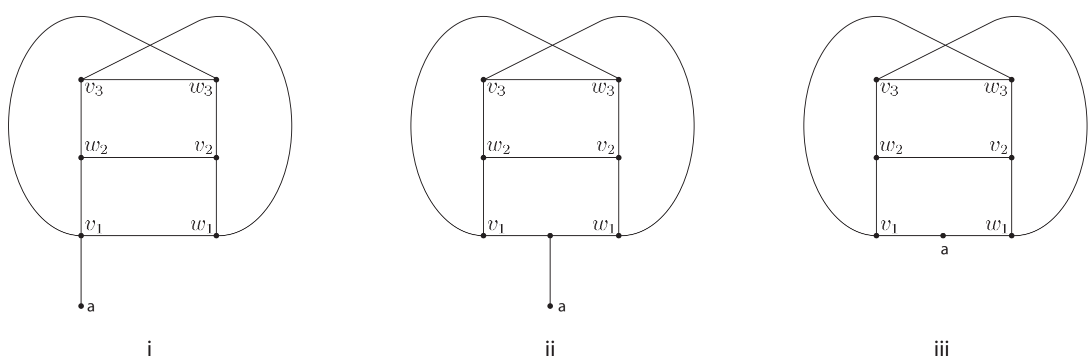

Figure 2.3. Smoothings of a split $K_{3,3}$ relative to a.

Definition 28. *Let G be a split $K_{3,3}$ and $a \in V(G)$. The* **smoothing of** G **relative to** a*, $G|_a$, is the graph formed by repeatedly deleting vertices of degree one (other than a) and smoothing vertices of degree two (other than a). Then either a is an original vertex or else $G|_a$ is one of the two graphs of Figure 2.3. In case $a = v_1$ is an original vertex of G, we say that v_1 is the* **nearest part** *of $K_{3,3}$ to a. In the case of Figure 2.3i or ii, we say that the edge v_1w_1 is the* **nearest part** *of $K_{3,3}$ to a.*

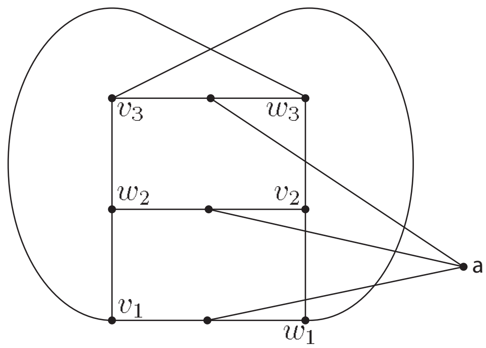

FIGURE 2.4. Adding a degree 3 vertex to a split $K_{3,3}$.

LEMMA 29. *If $G + a$ is formed by adding a vertex a of degree three to a split $K_{3,3}$ graph G and $G + a$ is not 1-apex, then $G + a$ is topologically equivalent to the graph of Figure 2.4.*

PROOF. By Lemma 25, there are paths from a to each original vertex that avoid all other original vertices. Let $N(a) = \{n_1, n_2, n_3\}$. As there are six vertices and $d(a) = 3$, then each n_i must have an edge of G as its nearest part, and up to relabeling of the original vertices, n_i has the edge $v_i w_i$ of G as its nearest part. This means $G + a$ is topologically equivalent to Figure 2.4. □

LEMMA 30. *If $G + a$ is formed by adding a vertex a of degree four to a split $K_{3,3}$ graph G and $G + a$ is not 1-apex, then $G + a$ is one of the seven graphs in Figure 2.5.*

PROOF. By Lemma 25, there are paths from a to each original vertex that avoid all other original vertices. Let $N(a) = \{n_1, n_2, n_3, n_4\}$. As there are six vertices and $d(a) = 4$, then there is an n_i, say n_1, that must have an edge of G as its nearest part. Since there are four original vertices left and three neighbors of a another n_j, say n_2, must have an edge of G as its nearest part, such that the original vertices that define share this edge are not the original vertices that share the edge near n_1. There are three graphs generated when a has a neighbor whose nearest part is an original vertex of G and four more when a has no such neighbor. Figure 2.5 shows the graphs that results from this condition. □

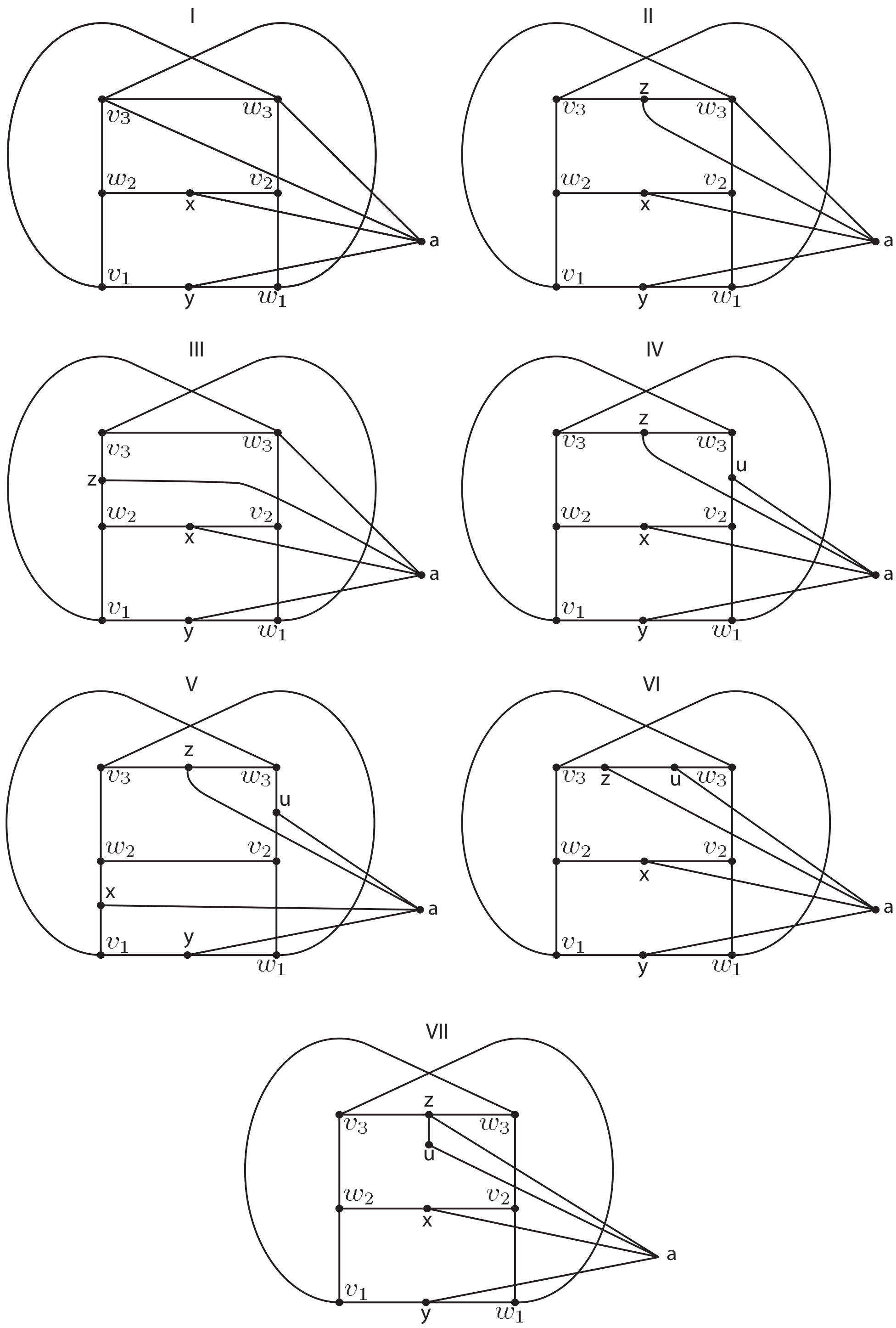

FIGURE 2.5. Adding a degree 4 vertex to a split $K_{3,3}$.

CHAPTER 3

Proof of Main Theorem

1. Intro to proof

THEOREM 31. *(Main Theorem) A graph G on 21 edges is MMIK if and only if it is one of the 14 descendants of K_7.*

We will break the following proof into cases by number of vertices. Let G be a MMIK graph of size 21. We can assume $\delta(G) \geq 3$, since deleting or smoothing v with $d(v) < 3$ from an IK graph will leave it IK. Since a $(15, 21)$ graph has at least one vertex of degree 2, we will assume $|G| \leq 14$. We start our argument with the case of $(14, 21)$ graphs and descend to $(13, 21)$ and so on.

This will allow us to consider only graphs that are triangle free. For instance, a MMIK (14,21) graph G that has a triangle admits a ΔY move to a connected $(15, 21)$ graph G^* that is IK. However, this implies $\delta(G^*) \leq 3$ which in turn means that G can be topologically simplified to a $(14, 20)$ graph. Such a graph is not IK (Lemma 10) and this contradicts our assumption that G was MMIK. Similarly, suppose that our main theorem has been verified for all $(14, 21)$ graphs. If G is a $(13, 21)$ graph that is MMIK and has a triangle, then applying ΔY to G gives us a connected IK graph, G^*, such that G^* has 14 vertices and 21 edges. Since G^* is connected, any minor of G^* will be obtained by the deletion of at least one edge. So any minor of G^* will have at most 20 edges, hence it will not be IK. Thus, G^* is MMIK and since we have established that the only MMIK $(14, 21)$ graphs are in the Heawood family, the MMIK $(13, 21)$ graphs that have triangles are precisely those which produce the established $(14, 21)$ MMIK graphs. That is, they are also Heawood graphs and so, our main theorem is verified in this case. The same argument can be applied to $(12, 21)$ graphs with a triangle once the main theorem has been established for $(13, 21)$ graphs, and so on.

In other words, having established the theorem for order $n + 1$, an MMIK graph G of order n that has a triangle must be one of the 14 Heawood graphs. We will start the argument for order n by assuming G is MMIK of that order. If we can show that G has a triangle, then we either have a contradiction (for example, if G does not have the degree sequence of a Heawood graph) or else we've succeeded in showing that G is Heawood. Either way, finding a triangle allows us to dispense with that case and move on to the next. For this reason, we won't always say explicitly whether the existence of a triangle amounts to a contradiction or an instance of a Heawood graph.

2. 14 vertex graphs

PROPOSITION 32. *Let G be a connected $(14, 21)$ graph. If G is not 2–apex, then G is the Heawood graph C_{14}.*

PROOF. Let G be a connected $(14, 21)$ graph and assume G is not 2–apex. Recall, if a 21 edge graph G has $\delta(G) < 3$ then G is topologically equivalent to a graph with fewer than 21 edges and is 2-apex by Lemma 10. Since $14 \times 3 = 2 \times 21$, G must have the degree sequence (3^{14}). For any vertex a, $G - a$ has degree sequence $(3^{10}, 2^3)$. Now choose another vertex, b, such that $G^* = G - a, b$ has the sequence $(3^6, 2^6)$. There are enough degree 3 vertices in $G - a$ to assure we can always choose such a b.

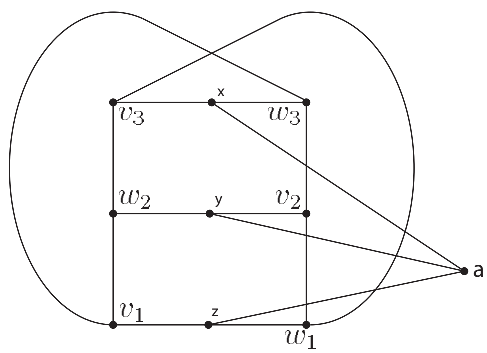

FIGURE 3.1. Adding a degree 3 vertex to a split K3,3 with vertices labeled.

Since G is not 2–apex and G^* has the sequence $(3^6, 2^6)$, then G^* must be topologically equivalent to $K_{3,3}$. By Lemma 29, $G^* + a$ must be topologically equivalent to Figure 2.4. Removing w_3 from $G^* + a$ will give us another graph, G', that is topologically equivalent to $K_{3,3}$. Again by Lemma 29 we see that b has paths to each original vertex in G' that contain no other original vertices of G' as well as having paths to the original vertices of G^* that contain no other original vertices of G^*. Hence, $G' + b$ is as in Figure 3.2. Adding w_3 back will give us C_{14}. Hence the only graph with degree sequence (3^{14}) that is not 2-apex is C_{14}.

□

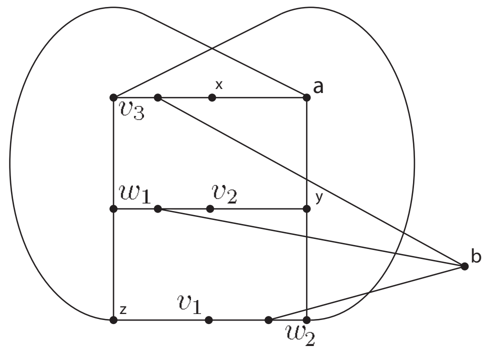

FIGURE 3.2. $G' + b$.

COROLLARY 33. *The only MMIK* $(14, 21)$ *graph is the Heawood graph* C_{14}.

3. 13 vertex graphs

PROPOSITION 34. *The only MMIK* $(13, 21)$ *graph is the Heawood graph* C_{13}.

PROOF. Notice that C_{13} has the degree sequence $3^{10}, 4^3$. Let G be an MMIK $(13, 21)$ graph.

An MMIK graph G will have $\delta(G) \geq 3$ and one of the following three degree sequences: $(3^{12}, 6)$, $(3^{11}, 4, 5)$, or $(3^{10}, 4^3)$.

Case 1: $(3^{12}, 6)$

Assume G has degree sequence $(3^{12}, 6)$. Remove a and b not adjacent with $d(a) = 6$, $d(b) = 3$. Then $\|G - a, b\| = 12$ and by [**M**] if $G - a, b$ is not planar, it has a K_2 component, which results in a triangle in G. So, there is no MMIK graph with this degree sequence.

Case 2: $(3^{11}, 4, 5)$

Assume G has degree sequence $(3^{11}, 4, 5)$. Remove the degree five and four vertices a and b. If a and b are not adjacent, then, as in the previous case, G has a triangle. So, we can assume a and b are adjacent. Then $\|G - a, b\| = 13$ and by [**M**] if $G - a, b$ is not planar, it is either $K_5 \cup K_2 \cup K_2 \cup K_2$, in which case G has a triangle, or has a component with $K_{3,3}$ minor as well as at least one tree component. However, a leaf of a tree component will form a triangle with a and b. So, there is no MMIK graph with this sequence.

Case 3: $(3^{10}, 4^3)$

Assume G has degree sequence $(3^{10}, 4^3)$. Remove two degree four vertices a and b. Assume a and b are not adjacent; then, $\|G - a, b\| = 13$ and by [**M**] if $G - a, b$ is not planar, it is either

$K_5 \cup K_2 \cup K_2 \cup K_2$, in which case G has a triangle, or has a component with $K_{3,3}$ minor as well as at least one tree component, T. If $|T| < 4$ then G has a triangle, so we'll assume $|T| > 3$ and we have two cases: $|T| = 4$ or $|T| = 5$.

If $|T| = 4$ and it is not a star, then, since a and b are adjacent to both the leaves of T, either a or b will form a triangle with a leaf of T and the vertex next to it. So we will assume T is a star. Since T has three leaves and a is adjacent to each leaf, a has one additional neighbor on the $K_{3,3}$ minor. The $K_{3,3}$ minor has seven vertices and ten edges, so it is a split $K_{3,3}$. Hence, removing one of its original vertices, v, will give us a planar graph, even when a is added back in. That is, $G - b, w$ is planar and G is 2-apex, hence not IK.

If $|T| = 5$ and T is not a star, then either a or b will form a triangle with one of the leaves of T and its adjacent vertex. So assume T is a star. Then, since T has four leaves, a has no neighbors on the $K_{3,3}$ component and neither does b. So G is not connected and thus not MMIK.

Say that a graph G has this sequence but there does not exist a pair of degree four vertices, a and b, such that a and b are not adjacent. Then G has a triangle, which means G is either C_{13} or not MMIK. This completes the argument for Case 3 and with it the proof of the proposition. □

4. 12 vertex graphs

PROPOSITION 35. *The only MMIK $(12, 21)$ graphs are the Heawood graphs C_{12} and H_{12}.*

PROOF. As in the previous cases, we take note that the degree sequences of C_{12} and H_{12} are $(3^7, 4^4, 5)$ and $(3^6, 4^6)$ respectively. Suppose G is a MMIK $(12, 21)$ graph. Our goal is to show that G is a Heawood graph as it then follows by Lemma 15 that G is C_{12} or H_{12}. As discussed in the introduction to this chapter, it suffices to show that G has a triangle. (Note that H_{12} has no triangles.)

Let us first consider a $(12, 21)$ MMIK graph G such that there exists a pair of vertices a and b with $\|G - a, b\| < 13$. By an Euler characteristic argument, if $G - a, b$ is nonplanar, then it contains at least one tree, T, and $|T| \leq 4$. Suppose $G - a, b$ contains a tree of order three or less, or a tree of order four that is not a star. Adding a and b back in will form a triangle on that tree, which implies G is either not MMIK or C_{12}. So we will assume that in the graph $G - a, b$, the tree component T has order four and is a star. This implies that the other component must be the graph $K_{3,3}$. Adding the vertex a back into the graph, we see that a needs to be adjacent to all the leaves of T. Also a must be adjacent to every vertex in the $K_{3,3}$ as otherwise the graph $G - b$ is 1–apex. Since $K_{3,3}$ is a split $K_{3,3}$, by Lemma 25, a must have a path to each original vertex that avoids any other original vertex. Hence $d(a) \geq 9$ and $\|G - b\| = 22$ which is impossible. So, if $G - a, b$ has size less than 13, G must be a Heawood graph.

This helps us narrow down the degree sequences we have to consider. For instance, suppose G is a $(12, 21)$ graph and has a vertex a, such that, $d(a) > 5$. Then there is another vertex b, such that, b is not adjacent to a and $d(b) > 2$, so $\|G - a, b\| < 13$. Hence, by the argument above, G is either C_{12} or H_{12} or not MMIK. Also, if there is a vertex of a of degree five and another vertex b such that $d(b) = 5$, or $d(b) = 4$ and b is not adjacent to a, then again $\|G - a, b\| < 13$ and G is either C_{12} or H_{12} or not MMIK. Recall that G MMIK implies $\delta(G) \geq 3$. In order to avoid a triangle among vertices of degree four or more, we need only consider the two cases where G has the degree sequence $(3^7, 4^4, 5)$ or $(3^6, 4^6)$.

Case 1: $(3^7, 4^4, 5)$

Assume G has degree sequence $(3^7, 4^4, 5)$. Denote the vertex of degree five as a and recall that it must be adjacent to all the vertices of degree four. Remove a and note that $G - a$ has the sequence $(2, 3^{10})$. Next, remove b such that the degree of b in G was four. Notice that if b is adjacent to the degree two vertex in $G - a$, that would imply a triangle in G, so we assume it is not. Then $G - a, b$ has the degree sequence $(2^4, 3^6)$. If $G - a, b$ is nonplanar then, since $\chi(G - a, b) = -3$, it contains a split $K_{3,3}$ and if it is not connected, its other component is a cycle of order 3 or 4. We notice something curious when we add a back in. Vertex a is adjacent to exactly one of the degree two vertices of $G - a, b$. Hence, in the sense of Lemma 25, a has paths to at most two original vertices on the nonplanar split $K_{3,3}$ of $G - a, b$ without having to pass any other original vertices, and a is also adjacent to b. The other three neighbors of a are original vertices of the split $K_{3,3}$. However, there are only three such neighbors and at least four other original vertices of the split $K_{3,3}$, to which a must be adjacent in order to form a path that avoids other original vertices. It follows from Lemma 25 that $G - b$ is one apex. Hence if G has the degree sequence $(3^7, 4^4, 5)$, then G is either a part of the Heawood family or it is not MMIK.

Case 2: $(3^6, 4^6)$

This case is hard as H_{12} has this degree sequence. Note that, unlike C_{12}, H_{12} has no triangles; we can eliminate many cases by showing they result in a triangle, but in the end we will need to explicitly show that a MMIK graph with this sequence is H_{12}. We will remove vertices a and b both of degree four, which we can assume to be nonadjacent. Let $G^* = G - a, b$. Notice that $\chi(G^*) = -3$ and we have two cases: either G^* is connected or it is not.

Assume that G^* is nonplanar and is not connected. If G^* contains a tree of order 3 or less, or if there is a tree with a leaf that has a neighbor of degree 2, then this will imply a triangle in G. So G^* is either a nonplanar $(7, 10)$ graph with a cycle of order three, a nonplanar $(6, 9)$ graph with a cycle of order four, or a nonplanar $(6, 10)$ graph with a star of order four. In the first case, a cycle of order 3 implies a triangle in G. If G^* has a star of order 4, then both vertices a and b are adjacent to each of the leaves of the star, so G is not connected, hence not MMIK. In the case where G^* has a cycle of order 4, denote it by C, then the other component is $K_{3,3}$. Since one of a and b has at least two neighbors on C, adding that vertex and removing any vertex of the $K_{3,3}$ in G^* results in a planar graph meaning G is 2–apex. Thus, we conclude that if G^* is not connected, then G is not MMIK.

We will now assume that G^* is connected. So by Lemma 22, G^* is a split $K_{3,3}$. Then by Lemma 30, we see that $G^* + a$ and $G^* + b$ are topologically equivalent to one of the seven graphs in Figure 2.5. We shall denote our graphs considered as the graphs listed in the figure and use the labels given to the vertices for convenience.

Notice that in the cases of VI and VII, $G^* + a$ has a triangle, since no vertex splits must be made, so we will assume that $G^* + a$ (and symmetrically $G^* + b$) is topologically equivalent to one of the other five graphs. If $G^* + a$ is topologically equivalent to V, then $|G^* + a| = 11$, so we do not have any extra vertex splits. Removing the vertices labeled z and x in the figure, it can be shown that the resulting graph has a planar representation. Furthermore, if b is not a neighbor of a, b can be a neighbor to all other remaining vertices and maintain the graph's planarity. Since our assumption was that a and be are not neighbors, we have shown that G is 2-apex in the case where $G^* + a$, and by symmetry $G^* + b$, is topologically equivalent to graph V in Figure 2.5.

Going on to the next possibility, we'll assume that $G^* + a$ is topologically equivalent to IV. Since $|G^* + a| = 11$ we do not have any vertex splits. If b is not a neighbor of y, then $G - v_1, w_1$ is planar. So now we assume that y and b are adjacent in G. If b is not adjacent to x or if b is not adjacent to z then $G - y, z$ and $G - y, x$ are planar respectively. Thus b will have x, y, and z as neighbors. If its fourth neighbor is not u, then G will have a triangle. This shows that both a and b will have x, y, z, and u as neighbors. So $G - y, x$ is planar. Since H_{12} does not have a triangle, if $G^* + a$ or $G^* + b$ is topologically equivalent to IV in Figure 2.5, then G is not MMIK

Considering the case where $G^* + a$ is topologically equivalent to to III in Figure 2.5, we notice that III has ten vertices and $G^* + a$ has eleven vertices. This implies that $G^* + a$ is III with a vertex split. We will denote the vertex created by this split u (and simply refer to u as the vertex split).

Notice that removing w_3 and z from $G^* + a$ gives us a planar graph, unless both a and b have u as a neighbor. Assume u is a neighbor of both a and b and recall that $G^* + b$ must be topologically equivalent to one of graph I, II, or III. We can rule out $G^* + b$ being topologically equivalent to II, since that would require another vertex split. We then see that in the graph $G^* + b$, the neighborhood of b, after smoothing u, is $\{x, y, z, w_3\}$, $\{x, y, v_3, w_3\}$, or $\{x, z, v_2, w_3\}$. In all of these cases, if we choose to remove x and w_3 we will get a planar graph even if we add a and b back in, since they both have u as a neighbor. Hence, in the case where $G^* + a$ or $G^* + b$ is topologically equivalent to III in Figure 2.5, G is 2-apex.

Now we will have $G^* + a$ be topologically equivalent to graph II in Figure 2.5. Notice, as when we considered graph III, there is a vertex split, u, on G^* not shown in II. In II, we see that the vertices z, a, and w_3 form a triangle, so we only must consider the graphs for which u is on one of the edges of this triangle. Assume u is between z and b. We see that u is a neighbor of both a and b. If $G^* + b$ is topologically equivalent to graph II, the neighborhood of b is $\{u, x, y, w_3\}$, $\{u, x, y, v_3\}$, or G contains a triangle. $G - w_3, v_3$ is planar in both options that do not have a triangle. So we assume that $G^* + b$ is topologically equivalent to graph I in Figure 2.5. Then, b is adjacent to u and u is adjacent to z, so b is adjacent to x or y. Without losing generality, we can say that b is adjacent to x. Hence, b also has w_1 and v_1 as neighbors. Clearly, $G - w_1, v_1$ is planar.

We shall now assume that u is between z and w_3. Consider again that u is adjacent to b. Not considering cases that would give us triangles, b has the neighborhood $\{u, y, x, v_3\}$ if $G^* + b$ is topologically equivalent to II in Figure 2.5, or b has the neighborhood $\{u, y, w_2, v_2\}$ or $\{u, x, w_1, v_1\}$ if $G^* + b$ is topologically equivalent to I. The graphs $G - w_3, v_3$, $G - w_2, v_2$, and $G - w_1, v_1$ are planar in each of these respective cases.

Next, suppose u is between w_3 and a. Notice that b is adjacent to u, so whether $G^* + b$ is topologically equivalent to graph I or II in Figure 2.5, b will have x and y as neighbors. Thus $G - w_3, v_3$ is planar. Since H_{12} does not have a triangle, if $G^* + a$ or $G^* + b$ is topologically equivalent to II in Figure 2.5, then G is not MMIK.

Lastly, we approach the case where $G^* + a$ is topologically equivalent to graph I in Figure 2.5. Notice again the triangle formed between a, w_3, and v_3, implies there is a vertex split, denote it by z, on one of the triangles edges. Obviously, the cases where z is between a and v_3 and between a and w_3 are symmetric. It is also symmetric if z is between v_3 and w_3, since we may have chose to remove v_3 instead of a and the graph isometric to I in Figure 2.5 relative to w_3 instead of a will be isomorphic to $G^* + a$ with z between a and v_3. Without loss of generality, we will assume that z is between v_3 and w_3.

We still have another vertex split, u, somewhere on our graph. If we remove v_3 and w_3, we notice that as long as both a and b are not adjacent to u, then the graph is planar. Vertex b is adjacent to z because z has degree 3 in G and b is also adjacent to u, which is is a neighbor of x or y since $G^* + b$ is topologically equivalent to I. In either case b is also adjacent to v_1 and w_1 or v_2 and w_2 respectively and the graph formed is H_{12}.

Therefore, if G is a MMIK $(12, 21)$, then it will either have a triangle, which will imply that it is the Heawood graph C_{12}, or it will be H_{12}.

□

5. 11 vertex graphs

Proposition 36. *The only MMIK* $(11, 21)$ *graphs are the Heawood graphs,* H_{11}*,* E_{11}*, and* C_{11}*.*

Proof. We begin the proof that H_{11}, E_{11}, and C_{11} are the only MMIK graphs on 11 vertices by noting that they have $(3^4, 4^5, 5^2)$, $(3^5, 4^4, 5^3)$, and $(3^4, 4^6, 6)$ as their respective degree sequences. Taking similar assumptions from the previous sections, we will use the fact that if an $(11, 21)$ graph G is MMIK, then it has a minimum degree of three. Having now established that, in the case of $(12, 21)$ graphs, the only MMIK graphs are the IK graphs of the Heawood Family, then we'll assume that any MMIK graph in the $(11, 21)$ case that has a triangle is an IK graph of the Heawood family. Hence, as in the previous cases, if G can be shown to have a triangle it is either not MMIK or it is a Heawood graph, so we are done with that case.

If an $(11, 21)$ graph, G, is 2-apex, then it is not IK. If not, then we will not be able to remove two vertices from it with the resulting graph being planar. So assume that we remove two vertices, a and b, and in the process we also remove at least 10 edges. The resulting graph has order 9 and has no more than 11 edges and a minimum degree of one. Thus $\chi(G-a, b) \geq -2$. Since $\chi(K_5) = -5$ then our graph cannot have a K_5 minor since it would require at least three trees and we do not have enough vertices. (Since $\delta(G - a, b) \geq 1$, a tree has at least two vertices.) If $G - a, b$ is non-planar it must have a $K_{3,3}$ minor. Now, $\chi(K_{3,3}) = -3$ so $G - a, b$ will have at least one tree of order two or three. If we attempt to add a and b back we see that a triangle will be formed in G because of the tree in $G_{a,b}$. Hence an $(11, 21)$ MMIK graph, G, is either a Heawood graph or any two vertices removed from G remove at most 9 edges.

The only degree sequences for which we cannot immediately remove ten edges with the removal of two vertices are $(6, 4^6, 3^4)$, $(5^4, 4, 3^6)$, $(5^3, 4^3, 3^5)$, $(5^2, 4^5, 3^4)$, $(5, 4^7, 3^3)$, and $(4^9, 3^2)$.

Case 1: $(6, 4^6, 3^4)$

Assume the graph G has $(6, 4^6, 3^4)$ as its degree sequence. Notice that if we cannot choose vertices a and b such that $G - a, b$ has 11 edges, then the vertex of degree 6 is a neighbor of each vertex of degree 4. Since it is impossible for no vertices having degree 4 in G to be mutually adjacent, there will be a triangle in G. Recall that C_{11} has such a degree sequence.

Case 2: $(5^4, 4, 3^6)$ and $(5^3, 4^3, 3^5)$

Assume the graph G has either the degree sequence $(5^4, 4, 3^6)$ or $(5^3, 4^3, 3^5)$. It's apparent that if we cannot remove an a and b from G such that $G - a, b$ has 11 edges, then all the vertices of degree 5 are mutual neighbors. Hence there is a triangle in G. Recall that E_{11}'s degree sequence is $(5^3, 4^3, 3^5)$.

Case 3: $(5^2, 4^5, 3^4)$ and $(5, 4^7, 3^3)$

Assume that G has either $(5^2, 4^5, 3^4)$ or $(5, 4^7, 3^3)$ as its degree sequence. We choose to remove two vertices a and b such that the degree of b is 5, the degree of a is 4, and b is not a neighbor of a. It may not be immediately obvious why we can choose such an a and b for the degree sequence $(5^2, 4^5, 3^4)$, however if b is a neighbor to all the vertices of degree 4, then the two vertices of degree 5 are not neighbors, so we can have a $(9, 11)$ graph with the removal of just two vertices. As discussed above, this leads to a triangle so that G is either not MMIK or Heawood.

So, we can remove vertices a and b that are not adjacent and of degree four and five. This means that $G - a, b$ is a $(9, 12)$ graph, so $\chi(G - a, b) = -3$. If $G - a, b$ is nonplanar and disconnected, then it has either a K_5 minor or a $K_{3,3}$ minor with at least one component of order at most three. Whether this component is a tree or a cycle does not matter since either way it will imply a triangle in G. So, we'll assume that $G - a, b$ is connected.

Denote $G - a, b$ as G^*. Since G^* is connected and $\chi(G^*) = -3$, if it is nonplanar then it has a $K_{3,3}$ minor, hence, by Lemma 22, G^* is a split $K_{3,3}$. Using Lemma 30 and the restriction that G has only 11 vertices, we see that $G^* + a$ is topologically equivalent one of the graphs I, II, or II in Figure 2.5. Notice that II automatically implies a triangle in G. If G^* is topologically equivalent to III, then removing v_1 and w_1, v_2 and w_2, or v_3 and w_3 respectively, shows us that b has y, x, and z as neighbors unless G is 2-apex. Since b is degree five and does not have a as a neighbor, then adding it back in will create a triangle in G.

If $G^* + a$ is topologically equivalent to I we notice that there must another vertex split, z, on one of the edges on the triangle formed by a, v_3, and w_3. If z is between v_3 and w_3 then $G - v_3, w_3$ is planar. If z is between a and v_3 or a and w_3 are symmetric cases, so we will assume z is between a and w_3. Since b is neighbor to z, if b has w_3 as a neighbor there is a triangle. If not, since any four of the other seven possible neighbors of b will have at least two neighboring vertices, hence G will have a triangle.

We conclude that if G has $(5^2, 4^5, 3^4)$ or $(5, 4^7, 3^3)$ as its degree sequence, then G will either be 2-apex or have a triangle. This means if G is MMIK with either of these sequences, then it is a graph of the Heawood family. H_{11} has the sequence $(5^2, 4^5, 3^4)$.

Case 4: $(4^9, 3^2)$

This degree sequence can be considered the hard case for $(11, 21)$ graphs since the maximum amount of edges we can take away with the removal of two vertices is 8. Such a graph has 9 vertices and 13 edges, of which there are many nonplanar graphs. So we will apply a slightly different method for this case. Assuming that G has the degree sequence $(4^9, 3^2)$ we first notice that together, the vertices of degree three have six neighbors. Hence, there is a vertex of degree four, denote it by v, whose neighbors are all vertices of degree four. If any of the neighbors of v are mutually adjacent, then G has a triangle. Removing all four neighbors of v gives us a graph $(7, 5)$, G^*, that has at least one vertex of degree zero. Also, since G has maximum degree four then G^* has maximum degree four. Since $\chi(G^*) = 2$ and G^* has at least one vertex of degree zero, then G^* is one of the following graphs with a degree zero vertex added to it. Either one of the four trees of order five and maximum degree four, a cycle of order five with a vertex of degree zero, a cycle of order four with a vertex split of degree one and a vertex of degree zero, or a cycle of order four with a tree of order two. Since a cycle of order three is a triangle, we exclude those cases. These graphs can be seen in Figure 3.3. Our goal is to show that we can add two vertices of the four we removed back on to each graph and keep its planarity.

Since the vertices we remove from G to make G^* all have v as a neighbor, each one will be a vertex of degree three on the graph $G^* - v$. Hence adding one of these vertices, call it a, back keeps the planarity of G^* and and it can be oriented such that is hides at most one vertex on $G^* - v$ from the outer face and such a vertex, call it u, will have a degree of two when a is added to G^*. Since we have three more vertices to choose to add, if all were neighbors of u, then u would have a degree of five in G, which contradicts our degree sequence assumption. So we will be able to add to vertices

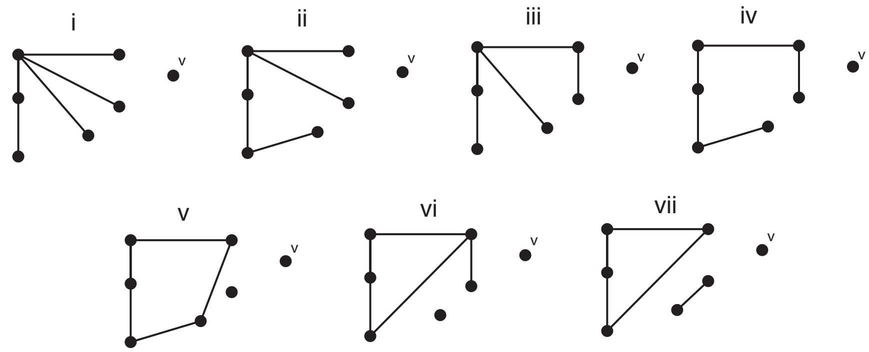

FIGURE 3.3. The seven $(7, 5)$ graphs with at least one degree zero vertex and a maximum degree of four.

back into our graph G^* while keeping its planarity. Hence if G has the degree sequence $(4^9, 3^2)$ and does not contain a triangle, then it will be 2-apex.

Since any degree sequence that an MMIK $(11, 21)$ graph can take will either contain a triangle or be 2-apex, then the only MMIK graphs with 21 edges and 11 vertices are IK graphs from the Heawood family, H_{11}, E_{11}, and C_{11}.

□

6. 10 vertex graphs

PROPOSITION 37. *The only MMIK* $(10, 21)$ *graphs are the Heawood graphs,* E_{10}*,* F_{10}*, and* H_{10}*.*

PROOF. Suppose G is a $(10, 21)$ graph. If we assume that G is MMIK, then we can assume it has minimum degree of 3. We have established that the only MMIK graphs on 11 vertices and 21 edges are the IK Heawood graphs, so we again can assume that a $(10, 21)$ graph that is MMIK and has a triangle is either E_{10}, F_{10}, and H_{10}. Consider if we can remove two vertices from G, a and b, such that $\|G - a, b\| \leq 10$. If $G - a, b$ is non-planar it has either a $K_{3,3}$ minor of a K_5 minor. Since $\chi(G - a, b) \leq -2$ and $\delta(G - a, b) \geq 1$, then $G - a, b$ will have a tree of order two or three and thus, there will be a triangle in G.

Similarly, we consider a graph G, where there are $a, b \in V(G)$, such that $\|G - a, b\| \leq 11$. According to Mattman there are eleven non-planar $(8, 11)$ graphs. So if G is non-planar it is one of the graphs in Figure 3.4.

With the exception of *iii* in Figure 3.4, which has a tree of order two as a component that implies a triangle in G, each graph is a split $K_{3,3}$ and in each graph of these graphs, there are at least two adjacent original vertices whose neighborhood is completely comprised of original vertices. Hence by Lemma 25, adding a (or b) back into $G - a, b$ will either result in a 1-apex graph or will create a triangle. So, G is either 2-apex or has a triangle. We now wish to consider cases where, for any $a, b \in V(G)$, $\|G - a, b\| \leq 12$.

The only degree sequences we now must consider are $(5^6, 3^4)$, $(5^5, 4^2, 3^3)$, $(5^4, 4^4, 3^2)$, $(5^3, 4^6, 3)$, and $(5^2, 4^8)$. For the first three sequences, we realize that if every vertex of degree five is mutually adjacent, then we have a triangle. If not, then we use the above argument. Thus, we are left with the $(5^2, 4^8)$ sequence.

We will now assume that G has the sequence $(5^2, 4^8)$ as well as assuming that G contains no triangles. We have considered the case where the two degree five vertices are not neighbors, since that would imply $a, b \in V(G)$, such that $\|G - a, b\| \leq 11$, so we will assume that $d(a) = d(b) = 5$ and that a and b are neighbors. This means that a and b do not have any mutual neighbors, since that would be a triangle. Hence, $G - a, b$ has the degree sequence (3^8) and it is a bipartite graph with one part comprised of the neighbors of a in G and the other comprised of the neighbors of b in G, or else it would also have a triangle. Constructing the 3-regular bipartite graph with eight vertices, we see in Figure 3.5 that it has a planar representation. Hence if a and b are neighbors then G is 2-apex.

We conclude that the their are no MMIK graphs with 21 edges and 10 vertices that do not have a triangle. Hence, $(10, 21)$ MMIK graphs are the IK Heawood graphs of order ten, E_{10}, F_{10}, and H_{10}.

□

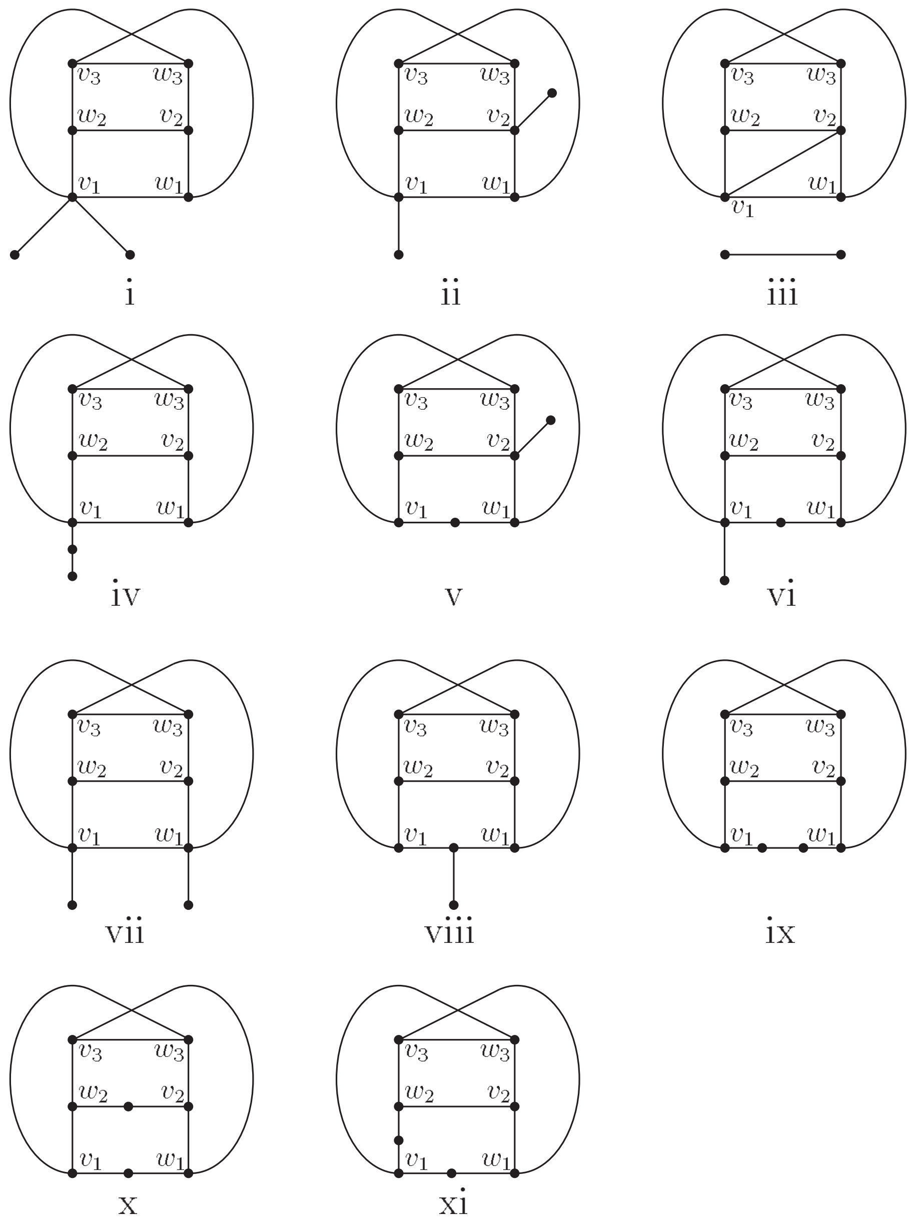

FIGURE 3.4. Non-planar graphs with eight vertices and eleven edges.

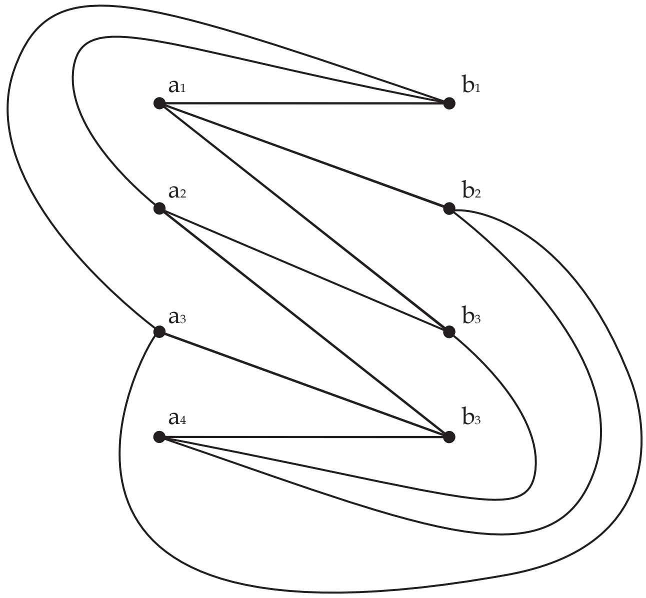

FIGURE 3.5. Planar representation of the 3-regular bipartite graph with eight vertices.

CHAPTER 4

Conclusion and Further Questions

It can be said that the 14 IK graphs of the Heawood family are the only MMIK graphs with 21 edges, thus proving the main theorem. It is also known that the other 6 graphs from the Heawood family are examples of non-IK graphs that are also not 2-apex. Are there any other examples with 21 edges? Though it is not included here, there is reason to suspect that the Y-Triangle move will preserve 2-apex as under the conditions that its applied to a graph G with 21 edges and 10 vertices or less. The question that follows from this is, under what circumstances will the Y-Triangle move preserve the 2-apex property? Is 2-apex preserved under Y-Triangle if G has 21 edges?

As mentioned in the Introduction, Lemma 25 provides a characterization for some 1-apex graphs. Figures 2.4 and 2.5 provide example of some of these obtained by this lemma. Figure 2.4 and graph I in Figure 2.5 are both graphs from the Petersen family and can be shown to be minor minimal intrinsically linked (MMIL). However, graphs III and V are not part of the Pertersen family but are both not 1-apex. Furthermore, by Lemma 25 it is not hard to see that any proper minor of these graph will be a 1-apex graph. So these graphs do not have any minors that are members of the Petersen family, which makes them good candidates for examples of graphs that are both not IL and minor minimal not-1-apex (MMNA). What are the other MMNA graphs?

CHAPTER 5

Aknowledgements

A huge thank you to my advisor Thomas Mattman. Without his insight, patience, and willingness to work with me this project would never have happened.

Bibliography

[BBFFHL] P. Blain, G. Bowlin, T. Fleming, J. Foisy, J. Hendricks, and J. LaCombe. 'Some Results on Intrinsically Knotted Graphs.' J. Knot Theory Ramifications, **16** (2007), 749–760.

[CG] J. Conway, C. McA. Gordon. 'Knots and links in spatial graphs.' J. Graph Theory, **7** (1983) 445–453.

[GMN] N. Goldberg, T.W. Mattman, R. Naimi. 'Many, many more intrinsically knotted graphs.' Preprint. (arXiv:1109.1632)

[KS] T. Kohara and S. Suzuki. 'Some remarks on knots and links in spatial graphs.' Knots 90, Osaka, 1990, de Gruyter (1992) 435–445.

[HNTY] R. Hanaki, R. Nikkuni, K. Taniyama and A. Yamazaki. 'On intrinsically knotted or completely 3-linked graphs.' Preprint. (arXiv:math.1006.0698)

[M] T.W. Mattman. 'Graphs of 20 edges are 2–apex, hence unknotted.' Alg. Geom. Top., **11** (2011) 691–718.. arxiv.org/0910.1575

[JKM] B. Johnson, M.E. Kidwell, and T.S. Michael 'Intrinsically Knotted Graphs Have at Least 21 Edges.' J. Knot Theory Ramifications, **19** (2010) 1423–1429.

[OT] M. Ozawa and Y. Tsutsumi. 'Primitive Spatial Graphs and Graph Minors.' Rev. Mat. Complut., **20** (2007), 391–406.

[RS] N. Robertson and P. Seymour. N. Robertson and P. Seymour. 'Graph minors. XX. Wagner's conjecture.' J. Combin. Theory Ser. B, **92**, (2004), 325–357.

[S] H. Sachs. 'On spatial representations of finite graphs.' Colloq. Math. Soc. János Bolyai (A. Hajnal, L. Lovasz, V.T. Sós, eds.), **37**, North Holland, Amsterdam, New York, 1984, 649–662.