2–bridge knot boundary slopes: Diameter and Genus

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ABSTRACT. We prove that for 2–bridge knots, the diameter, D, of the set of boundary slopes is twice the crossing number, c. This constitutes part of a proof that, for all Montesinos knots in S^3, $D \leq 2c$. In addition, we characterize the 2–bridge knots with four or fewer boundary slopes and show that they each have a boundary slope of genus two or less. We also present examples of knots in S^3 with $D > 2c$. We propose questions that explore a possible connection between the number of boundary slopes and slopes of small genus.

Acknowledgments

Contents

CHAPTER 1

Introduction

In this chapter, we give an introduction to 2–bridge or rational knots, crossing numbers, and boundary slopes. We will conclude this chapter with an overview of the thesis.

1. Introduction to Knot Theory

This section will contain a brief introduction to knot theory. For a more detailed discussion, see [1].

In colloquial terms, a *knot* is any closed loop in space. This can be imagined as a rope or cord which has been tangled and looped around itself, and then had its two loose ends attached to each other. See Figure 1.1 for an example of a knot.
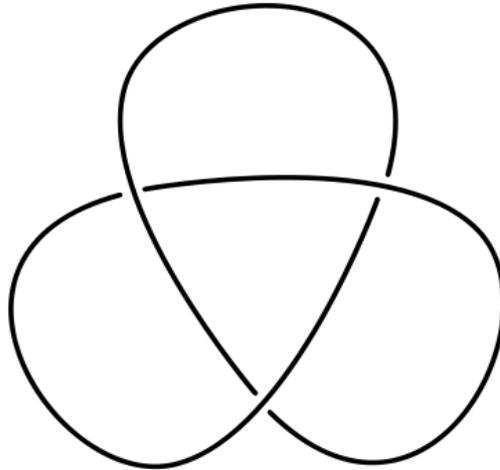
FIGURE 1.1. A Trefoil knot.

As long as we don't break the rope or detach the ends, then we can perform any contortions or further tanglings on the knot that we wish, and we will say that we still have the same knot. A *projection* of a knot is a 2–dimensional representation of a knot. For a given projection of a knot, each time the "arcs" of the knot cross over each other, they form a crossing with an overarc and an underarc. See Figure 1.2 for an example of transforming a projection of the Figure–8 knot into another

projection of the same knot. Note that, though the knot appears very similar, overarcs and underarcs have been swapped at each crossing.
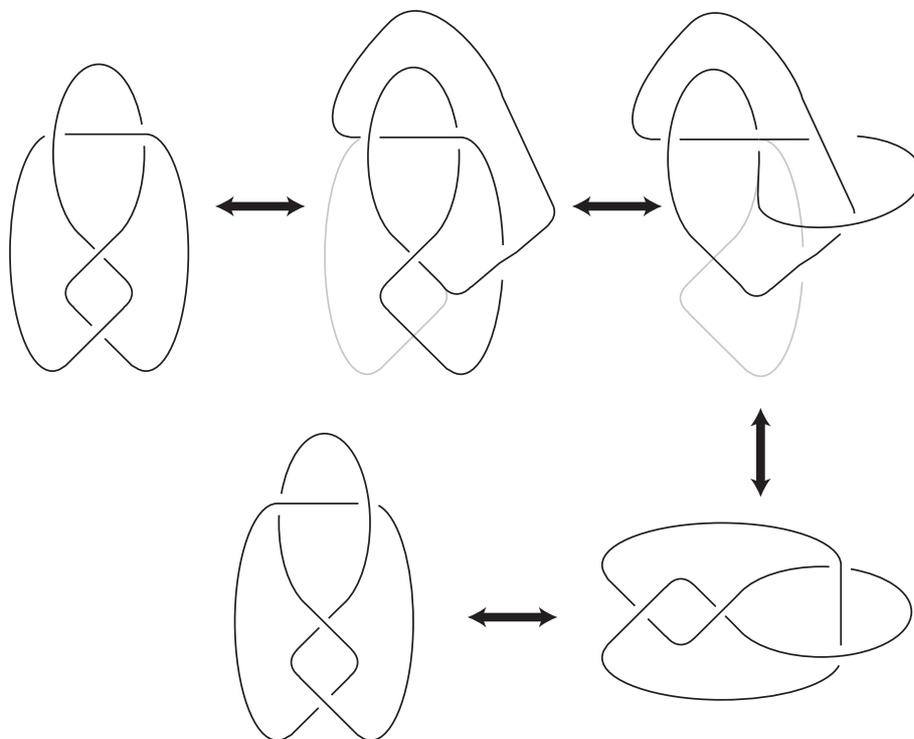
FIGURE 1.2. Transforming a projection of the Figure–8 knot with 4 crossings, into another projection.

2. Tangles

A *tangle*, as defined in [1], is a region within a knot projection where we can draw a circle which intersects the knot in precisely four places (Figure 1.3). The four points of intersection are said to be at the four compass directions of NW, NE, SW, and SE. The next section will describe rational tangles. All of the tanlges in Figure 1.3 are rational tangles.
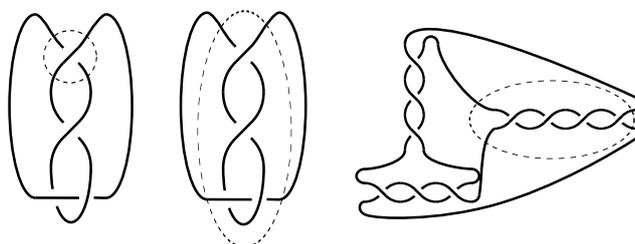
FIGURE 1.3. Three examples of tangles.

3. Rational Knots

Rational knots, also called 2–*bridge knots*, are knots which can be drawn in the form of Figure 1.4. These are called 2–bridge knots because they can be represented as in the figure with exactly two local maxima or "bridges" as at the top of the figure. They are called rational knots because of a connection with rational numbers, which will be discussed later.
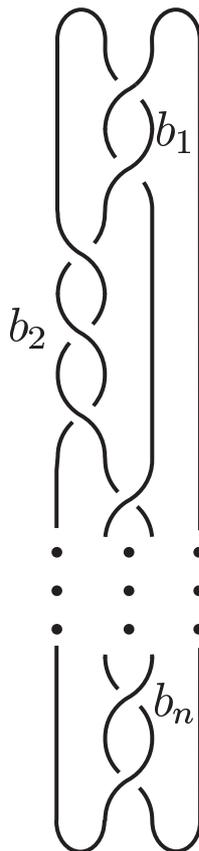
Figure 1.4. The general form of a 2–bridge knot.

John H. Conway [2] studied these and other types of knots, and developed a notation which we call the *Conway notation* for describing these knots. The Conway notation is developed from a slightly different representation of rational knots than that of the above figure. I will borrow on the description in [1] of how to form *rational tangles*. First, we must define "positive" and "negative" twists. A "positive" twist is a twist in which the overarc passes from SW to NE and the underarc passes from NW to SE, as in Figure 1.5.

We begin with the so-called 0 tangle, which consists of two horizontal strings. We then "twist" the two strings at right together to form b_1 crossings, where a positive b_1 indicates positive crossings and negative b_1 indicates negative crossings. Next, we reflect the tangle about the NW to SE line, and then twist the two
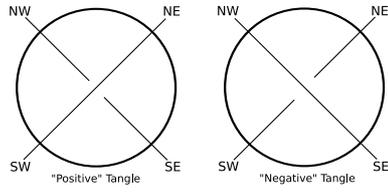
FIGURE 1.5. "Positive" and "Negative" twists.

rightmost strands b_2 times. We continue reflecting across the NW to SE line and twisting b_i times. Then we connect the bottom strands to each other, and the top strands to each other. The b_i sequence we've generated is the Conway notation of this knot. See Figure 1.6 for an example of forming the 3 3 4 rational knot.
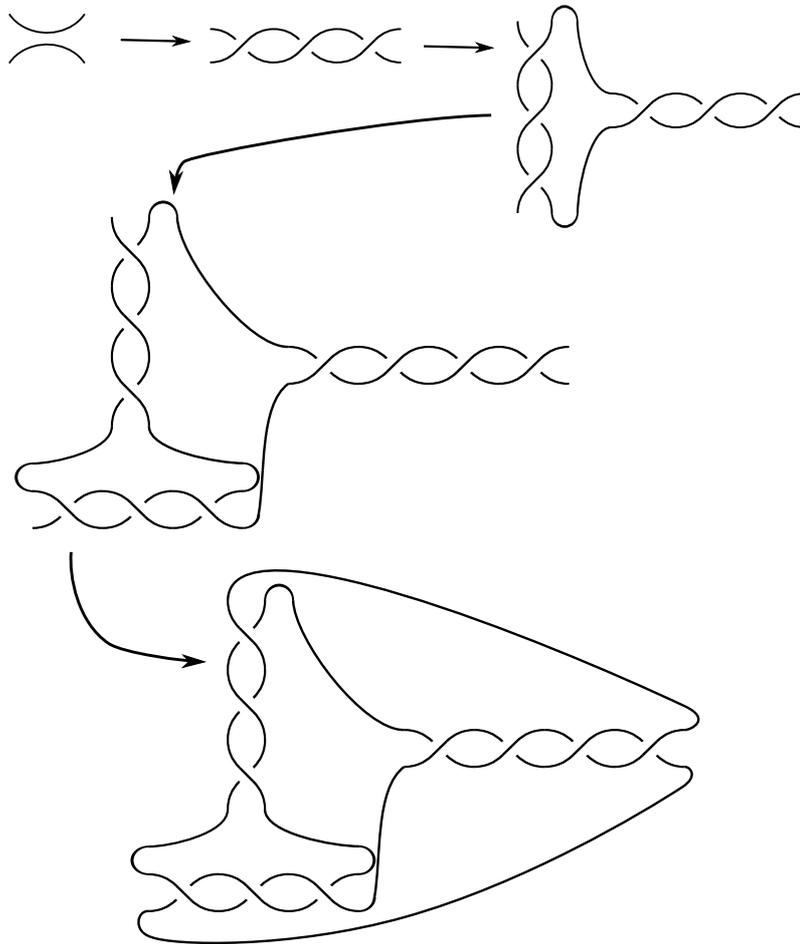
FIGURE 1.6. Forming of the 3 3 4 rational tangle and knot.

Interestingly, two Conway notations may represent the same underlying knot. For example, the 3 3 4 knot described above can also be obtained from the Conway notation -2 2 -5 2 -2 2. This will be discussed in more detail later.

4. Crossing Number

Recall that a knot crossing in a projection is a place where the knot crosses itself. The crossing number of a knot K, denote $c(K)$, is defined as the minimum number of crossings among all projections of the knot. For an example, see Figures 1.7 and 1.8.
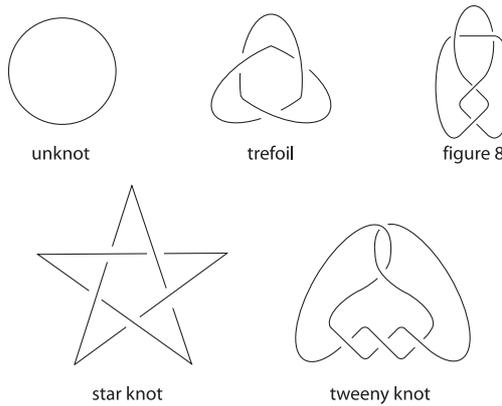
FIGURE 1.7. From top left to lower right, $c(K) = 0, 3, 4, 5, 5$
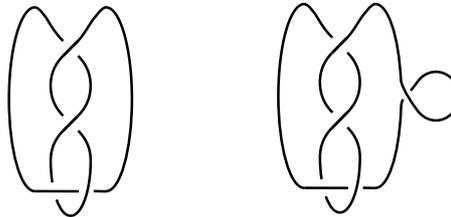
FIGURE 1.8. $c(K) = 4$ for both projections of the Figure–8 knot

An *alternating knot projection* is a knot projection in which, if we travel along the knot, we alternate between over-crossings and under-crossings.

A *reduced knot projection* is a knot projection in which no crossings are reducible, where a reducible crossing is defined as a crossing for which there exists a circle which intersects the crossing once, but does not intersect the knot anywhere else. *Reducible crossings* are those that can be removed by a single twist. Thistlethwaite [21], Kauffman [9] and Murasugi [16][17] independently proved the first Tait conjecture, which states that a reduced alternating projection of a knot yields the crossing number.

In general, determining the crossing number of a knot can be quite difficult. In fact, even with rational knots where we know the Conway notation, it's not immediately apparent how to determine the crossing number. Every rational knot has a

Conway notation which consists of all positive integers, as we will see below. It is evident that such an "all-positive" Conway notation produces a reduced, alternating projection, and by the first Tait conjecture, the crossing number for a rational knot is simply the sum of the terms of this "all-positive" Conway notation [10][4]. It is then a question of determining this all-positive Conway notation, and this is where the connection between rational knots and rational numbers comes in. To describe this connection, we will first introduce the notion of continued fractions.

5. Continued Fractions

A *continued fraction expansion* of $\frac{p}{q}$ is a fraction of the form

$$\frac{p}{q} = c + \cfrac{1}{b_0 + \cfrac{1}{b_1 + \cfrac{1}{\cdots + \frac{1}{b_n}}}} = [c; b_0, b_1, \ldots, b_n],$$

where $c \in \mathbb{Z}$ and each b_i, for $0 \leq i \leq n$, is a nonzero integer. The b_i are called *partial quotients* or *terms*. The *simple continued fraction* of $\frac{p}{q}$ is the unique one having all terms positive. We will also require $b_n > 1$ for all simple continued fractions where $p/q \notin \mathbb{Z}$ for the sake of uniqueness, since $[c; b_0, b_1, \ldots, b_{n-1}, 1] = [c; b_0, b_1, \ldots, b_{n-1} + 1]$.

Continued fractions are related to rational knots in surprisingly direct way. Given the Conway notation $b_1 b_2 \ldots b_n$ for a rational knot K, we can calculate a continued fraction by reversing the Conway notation: $[0; b_n, b_{n-1}, \ldots, b_2, b_1] = \frac{p}{q}$. We then denote this knot as $K(p/q)$. Schubert [19] showed that two rational knots $K(p/q)$ and $K(p'/q')$ are equivalent if and only if $q = q'$ and $p' \equiv p^{\pm 1} \bmod q$. Hence, for any given knot $K(p'/q)$, there exists an equivalent knot $K(p/q)$ where $0 \leq p < q$ (i.e. $0 \leq p/q < 1$). It follows that every rational knot has some projection in which the Conway notation consists only of positive integers, which leads us to the crossing number of the knot.

6. Essential Surfaces

In this section, we will provide an informal introduction to essential surfaces for knots, and describe the algorithm for producing Seifert surfaces. For more details, see [11]. Essential surfaces are surfaces whose boundary is (possibly several parallel copies of) a given knot. Although essential surfaces are, in general, difficult to visualize, Herbert Seifert discovered a simple algorithm for constructing an oriented surface from a knot projection. We call these surfaces *Seifert surfaces*. They are examples of essential surfaces and the following is a description of the algorithm.

To produce a Seifert surface for a particular knot and projection of that knot, we first assign an orientation to the projection. See figure 1.9 for an example.
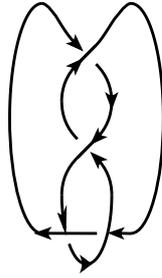
FIGURE 1.9. An oriented Figure–8 knot.

At each crossing of the knot are four strands in total. Due to the orientation, two strands will be "incoming" which will be connected to two "outgoing" strands. We then "cross-connect" each incoming strand with the outgoing strand adjacent to it – that is, the outgoing strand to which it was not previously connected. By doing this, we break the link into a set of oriented discs (Figure 1.10).
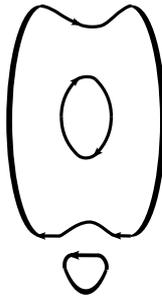
FIGURE 1.10. Incoming and outgoing strands cross-connected.

Lastly, connect each of the discs to discs they were originally connected to using a band with a half-twist in the same manner that they were originaly crossing. When done, the surface will look very much like the original knot. Indeed, the boundary of this surface *is* the orginal knot. And the Seifert surface will be oriented (that is, it will have two distinct sides) (Figure 1.11).
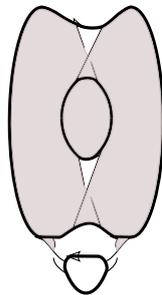
FIGURE 1.11. Oriented Seifert surface for the Figure–8 knot.

Essential surfaces raise the notion of *boundary slopes*, which describe the way essential surfaces intersect the boundary torus, which is simply a thickened version of the knot. Boundary slopes are pairs of integers, often represented as either an ordered pair or, as throughout this paper, a rational number. In general, essential surfaces and boundary slopes are not easy to determine. However, for 2–bridge knots, it is known that the boundary slopes will always be even integers. Furthermore, there is a relatively simple method for computing the boundary slopes, which we describe in the following section.

7. Boundary Slopes of 2–Bridge Knots

In this section we review Hatcher and Thurston's [**5**] method for computing the boundary slopes of a 2–bridge knot. Let $K\left(\frac{p}{q}\right)$ denote the 2–bridge knot associated to the fraction $\frac{p}{q}$. Recall that the equivalence relationship between rational knots allows us to assume that $0 \leq \frac{p}{q} < 1$. As $\frac{p}{q} = 0$ corresponds to the unknot, we will often further assume that $0 < \frac{p}{q} < 1$.

The various types of boundary surfaces, and the slopes at which they intersect the knot, are difficult to imagine. However, with rational knots, it is very easy to calculate the boundary slopes. Following [**5**], to calculate the boundary slopes associated with a rational knot, we must first determine all continued fraction representations of that knot where $|b_i| \geq 2$ for each i. These are called the *boundary slope continued fractions*. Among the boundary slope continued fractions, there will always be a unique one having all b_i of even parity, which we call the *longitude* continued fraction. This is the slope of the Seifert surface described in the last section.

We then, for each boundary slope continued fraction, compare the partial quotients to the pattern $[+ - + - \cdots]$. The number of terms matching this pattern we call n^+, and the number of terms not matching this pattern (i.e., the total number of terms minus n^+) we call n^-; note that these terms match the pattern $[- + - + \cdots]$.

In this way, we associate to each continued fraction two non-negative integers n^+ and n^-. The boundary slope is then given by comparing the difference $n^+ - n^-$ with that corresponding to the longitude: $n_0^+ - n_0^-$; the boundary slope associated with the continued fraction is $2\left[(n^+ - n^-) - \left(n_0^+ - n_0^-\right)\right]$. Applying this calculation to every boundary slope continued fraction gives the set of boundary slopes $B(K) = B$. B is a finite set of even integers. The diameter $D(K) = D$ is the difference between the biggest and smallest elements of B.

8. Example of Calculating Boundary Slopes

In this section, we provide an example of calculating the boundary slopes for $K(10/34)$, the knot formed by Conway notation 3 3 4. For an illustration of this knot, see Figure 1.12.
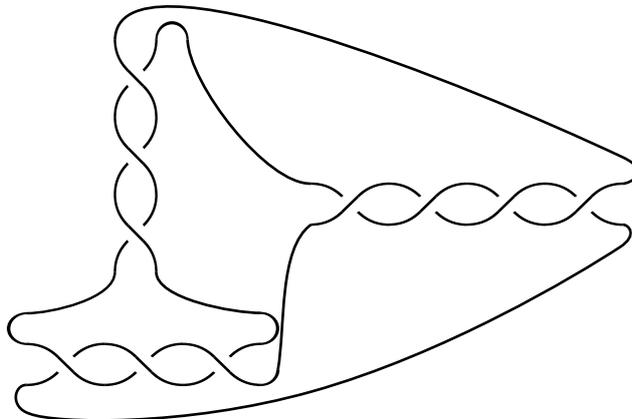
FIGURE 1.12. $K(10/34)$

The longitude continued fraction for this knot is $10/34 = [0; 4, 4, -2, 2]$. The other boundary slope continued fractions can be calculated as the simple continued fraction $[0; 4, 3, 3]$, plus $[0, 5, -2, 2, -4]$, $[1; -2, 2, -2, 4, 3]$, and $[1; -2, 2, -2, 5, -2, 2]$. Below, we list these continued fractions with their associated n^+ and n^-.

$$
\begin{array}{llll}
[0; 4, 4, -2, 2] & \Rightarrow & n_0^+ = 1 & n_0^- = 3 \\
[0; 4, 3, 3] & \Rightarrow & n^+ = 2 & n^- = 1 \\
[0, 5, -2, 2, -4] & \Rightarrow & n^+ = 4 & n^- = 0 \\
[1; -2, 2, -2, 4, 3] & \Rightarrow & n^+ = 1 & n^- = 4 \\
[1; -2, 2, -2, 5, -2, 2] & \Rightarrow & n^+ = 0 & n^- = 6
\end{array}
$$

We now calculate the set of boundary slopes from the above data using $2\left[(n^+ - n^-) - \left(n_0^+ - n_0^-\right)\right]$:

$$
\begin{array}{llll}
n^+ = 1 & n^- = 3 & \Rightarrow & 2\left[(1-3) - (1-3)\right] = 0 \\
n^+ = 2 & n^- = 1 & \Rightarrow & 2\left[(2-1) - (1-3)\right] = 6 \\
n^+ = 4 & n^- = 0 & \Rightarrow & 2\left[(4-0) - (1-3)\right] = 12 \\
n^+ = 1 & n^- = 4 & \Rightarrow & 2\left[(1-4) - (1-3)\right] = -2 \\
n^+ = 0 & n^- = 6 & \Rightarrow & 2\left[(0-6) - (1-3)\right] = -8
\end{array}
$$

Therefore, we get $D(K(10/34)) = [-8, -2, 0, 6, 12]$.

9. Results

Ichihara [6] told us of a conjecture; he and Mizushima have since proved [7] the conjecture, though they refer to our result for part of the proof. Let K be a Montessinos knot in S^3. Let $D(K)$ denote the diameter of the set of boundary slopes of K, and let $c(K)$ be the crossing number of K.

THEOREM. *For K a Montesinos knot in S^3, $D(K) \leq 2c(K)$.*

Since 0, being the slope of a Seifert surface, is always included in the set of boundary slopes, we have, as an immediate consequence, a conjecture due to Ishikawa and Shimokawa [8]:

THEOREM. *Let b be a finite boundary slope for K a Montesinos knot in S^3. Then $|b| \leq 2c(K)$.*

It is easy to verify this result for the torus knots. For the unknot, $D(K) = 0 = 2c(K)$. For a non-trivial torus knot K of type (p, q) we can assume p, q relatively prime with $2 \leq q < p$. The boundary slopes are 0 and pq [**15, 22**] while the crossing number is $c(K) = pq - p$ [**18**]. Thus, $D(K) = pq \leq pq + p(q-2) = 2c(K)$. Moreover, we have equality for the torus 2–bridge knots, which are of the form $(p, 2)$ with p odd.

We will show that this equality obtains for all 2–bridge knots, and hence all Montesinos knots with fewer than 3 components:

10. Map of Thesis

In Chapter 1, we provided an introduction to knot theory, continued fractions, and the basic tools required to understand the rest of this paper, and stated our results. Chapter 2 will contain several lemmas and theorems, culminating in the proof of our results, and several corollaries. It will also contain proofs of several corollaries.

THEOREM 1. *For K a 2–bridge knot, $D(K) = 2c(K)$.*

COROLLARY 1. *Let b be a boundary slope for a 2–bridge knot K. Then $|b| \leq 2c(K)$.*

THEOREM 2. *The boundary slope continued fractions of $K(p/q)$ are among the continued fractions obtained by applying substitutions at non-adjacent positions in the simple continued fraction of p/q.*

COROLLARY 2. *If $\frac{p}{q} = [0, a_0, a_1, \ldots, a_n]$ is a simple continued fraction, then $K(p/q)$ has at most F_{n+2} boundary slopes where F_n is the nth Fibonacci number.*

THEOREM 3. *Let $K = K(p/q)$ be a 2–bridge knot.*
- *If K has only two distinct boundary slopes, then K is a torus knot and $p = 1$ or $p = q - 1$.*
- *If K has precisely three boundary slopes, then $p|(q \pm 1)$ or $(q - p)|(q \pm 1)$.*
- *If K has precisely four boundary slopes, then one of the following holds: $p|(q + 1)$, $(q - p)|(q + 1)$, $(p \pm 1)|q$, or $(q - p \pm 1)|q$.*

Chapter 3 will then discuss open questions pertaining to our results.

CHAPTER 2

Proving Results

In this chapter, we will provide the theorems and their proofs. We begin by stating our primary result, and proceed by proving prerequisite results. We will then prove Theorem 1 and several corollaries and supplementary results.

THEOREM 1. *For K a 2–bridge knot, $D(K) = 2c(K)$.*

COROLLARY 1. *Let b be a boundary slope for a 2–bridge knot K. Then $|b| \leq 2c(K)$.*

This bound is sharp for the $(p, 2)$ torus knots and there are many examples showing that it is also sharp for hyperbolic 2–bridge knots. Such examples are given by 2–bridge knots that are "checkerboard;" an alternating knot K is called checkerboard if it possesses a reduced alternating diagram such that one of the checkerboard surfaces is an essential Seifert surface for K. In this case, the boundary slope b of the other checkerboard surface satisfies the equality in Corollary 1.

In Section 2 of this chapter, we present two substitution rules for continued fractions. These substitution rules will allow us to produce all possible boundary slope continued fractions for a given rational number:

THEOREM 2. *The boundary slope continued fractions of $K(p/q)$ are among the continued fractions obtained by applying substitutions at non-adjacent positions in the simple continued fraction of p/q.*

The proof of Theorem 2 is presented in Section 3 along with the following corollary, originally proved by Hatcher and Thurston [**5**].

COROLLARY 2. *If $\frac{p}{q} = [0, a_0, a_1, \ldots, a_n]$ is a simple continued fraction, then $K(p/q)$ has at most F_{n+2} boundary slopes where F_n is the nth Fibonacci number.*

In Section 4 we show how to compare the boundary slopes obtained from different substitution patterns. This allows us to identify the patterns corresponding to the maximum and minimum boundary slopes and thereby to prove Theorem 1.

In Section 5 we characterize the 2–bridge knots that have no more than four boundary slopes.

THEOREM 3. *Let $K = K(p/q)$ be a 2–bridge knot.*
- *If K has only two distinct boundary slopes, then K is a torus knot and $p = 1$ or $p = q - 1$.*
- *If K has precisely three boundary slopes, then $p|(q \pm 1)$ or $(q - p)|(q \pm 1)$.*
- *If K has precisely four boundary slopes, then one of the following holds: $p|(q + 1)$, $(q - p)|(q + 1)$, $(p \pm 1)|q$, or $(q - p \pm 1)|q$.*

Note that the torus knots are also the only 2–bridge knots with a genus 0 boundary slope (see [**5**, Theorem 2(a)]). Thus, the set of 2–bridge knots admitting a genus 0 boundary slope exactly coincides with those having two boundary slopes.

The situation for genus 1 and 2 boundary slopes is similar. Using [5] (see also [20]) the genus of a k-sheeted surface carried by a continued fraction $[0, b_0, b_1, \ldots b_n]$ is $g = (2 + k(n-1))/2$ which is 1 only if $n = 1$. In other words, the 2–bridge knots having a genus 1 boundary slope are exactly the trefoil knot along with the hyperbolic knots for which $p \mid (q \pm 1)$ or $(q - p) \mid (q \pm 1)$. Thus, if a 2–bridge knot has exactly three boundary slopes, then it has a genus 1 boundary slope. Similarly, if $K(p/q)$ has exactly four boundary slopes, then it has a boundary slope of genus 2 or less.

The converses of these statements are not quite true. For example, a knot with a genus 1 boundary slope may have four boundary slopes (and not just three), e.g., $K(4/11)$ has boundary slopes -4, 0, 2, 8, the last being of genus 1. Still, this suggests the following:

Question: If the knot K has a boundary slope of small genus, does it follow that K has few boundary slopes? Conversely, do few boundary slopes imply a slope of small genus?

That is, does the pattern we observe for 2–bridge knots persist beyond genus 2? What if we consider more general classes of knots?

1. Continued Fraction Identities

In this section, we provide two continued fraction identities and their straight-forward proofs. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$. We will use the notation $(b_0, \ldots, b_m)^c$ to mean that the pattern "$b_0, \ldots, b_m$" is repeated c times, with c being any nonnegative integer, e.g., $[0, (-2, 2)^2] = [0, -2, 2, -2, 2]$ and $[0, (-2, 2)^0, 2] = [0, 2]$. Note that $[b_0, \ldots, b_m, p/q] = [b_0, \ldots, b_m, a_0, \ldots, a_n]$ where $p/q = [a_0, \ldots, a_n]$.

IDENTITY 1. *Let $c \in \mathbb{N}_0$ and $k \in \mathbb{Q}^*$. Then*

$$[(-2, 2)^c, k] = \frac{2ck + 2c + k}{1 - 2ck - 2c}$$

Note that the denominator becomes zero only in the case where the continued fraction does not represent a rational number.

PROOF. We proceed by induction on c.
Base Case $(c = 0)$: $[k] = k = \frac{2 \cdot 0 \cdot k + 2 \cdot 0 + k}{1 - 2 \cdot 0 \cdot k - 2 \cdot 0}$.
Induction Step: Assume that $[(-2, 2)^c, k] = \frac{2ck + 2c + k}{1 - 2ck - 2c}$. Then

$$
\begin{aligned}
[(-2, 2)^{c+1}, k] &= -2 + \cfrac{1}{2 + \cfrac{1}{[(-2,2)^c, k]}} \\
&= -2 + \cfrac{1}{2 + \cfrac{1}{\frac{2ck + 2c + k}{1 - 2ck - 2c}}} \\
&= -2 + \cfrac{2ck + 2c + k}{2cl + 2c + 2k + 1} \\
&= \frac{-2ck - 2c - 3k - 2}{2ck + 2c + 2k + 1} \\
&= \frac{2(c+1)k + 2(c+1) + k}{1 - 2(c+1)k - 2(c+1)}
\end{aligned}
$$

$\square$

IDENTITY 2. *Let $c \in \mathbb{N}_0$ and $k \in \mathbb{Q}^*$. Then*

$$[(2, -2)^c, k] = \frac{2ck - 2c + k}{2ck - 2c + 1}$$

Again, note that the denominator becomes zero only in the case where the continued fraction does not represent a rational number.

PROOF. We proceed with two cases based on c.

Case 1 ($c = 0$): $[k] = k = \frac{2 \cdot 0 \cdot k - 2 \cdot 0 + k}{2 \cdot 0 \cdot k - 2 \cdot 0 + 1}$

Case 2 ($c > 0$): Note that $[(2, -2)^c, k] = [2, (-2, 2)^{c-1}, -2, k]$. Apply Identity 1. $\qquad \square$

2. Continued Fraction Substitution Rules

In this section, we will prove four identities, or substitution rules, which will be used to derive equal continued fractions. As we will illustrate at the end of the section, these substitutions can be used to derive all the boundary slope continued fractions of $K(p/q)$ from the simple continued fraction of p/q. We conclude the section with an example to illustrate how these rules can be applied to a specific continued fraction.

Throughout this section, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$.

SUBSTITUTION 1. *Let $n \in \mathbb{N}$. Let $b_0 \in \mathbb{Z}$ and $b_1 \in \mathbb{N}$. If $n = 2$ then let $b_2 \in \mathbb{Z} \setminus \{0, -1\}$. If $n \geq 3$ then let $b_i \in \mathbb{Z}^*$ for all $2 \leq i \leq n$. Then $[b_0, 2b_1, b_2, b_3, \ldots, b_n] = [b_0 + 1, (-2, 2)^{b_1 - 1}, -2, b_2 + 1, b_3, b_4, \ldots, b_n]$. In particular, $[b_0, 2b_1, b_2] = [b_0 + 1, (-2, 2)^{b_1 - 1}, -2, b_2 + 1]$.*

PROOF. We will prove this substitution rule in three parts.

Case 1 ($n = 1$): We want to show that $[b_0, 2b_1] = [b_0 + 1, (-2, 2)^{b_1 - 1}, -2]$.

$$[b_0 + 1, (-2, 2)^{b_1 - 1}, -2]$$
$$= b_0 + 1 + \frac{1}{\frac{2(b_1 - 1)(-2) + 2(b_1 - 1) + (-2)}{1 - 2(b_1 - 1)(-2) - 2(b_1 - 1)}} \quad \text{(Apply Identity 1)}$$
$$= b_0 + 1 + \frac{-2b_1 + 1}{2b_1}$$
$$= b_0 + \frac{1}{2b_1}$$
$$= [b_0, 2b_1]$$

Case 2 ($n = 2$): We want to show that $[b_0, 2b_1, b_2] = [b_0 + 1, (-2, 2)^{b_1 - 1}, -2, b_2 + 1]$.

$$[b_0 + 1, (-2, 2)^{b_1 - 1}, -2, b_2 + 1]$$
$$= b_0 + 1 + \frac{1}{\frac{2(b_1 - 1)(\frac{-2b_2 - 1}{b_2 + 1}) + 2(b_1 - 1) + \frac{-2b_2 - 1}{b_2 + 1}}{1 - 2(b_1 - 1)\frac{-2b_2 - 1}{b_2 + 1} - 2(b_1 - 1)}} \quad \text{(Apply Identity 1)}$$
$$= b_0 + 1 + \frac{b_2 - 2b_1 b_2 - 1}{2b_1 b_2 + 1}$$
$$= b_0 + \frac{1}{\frac{2b_1 b_2 + 1}{b_2}}$$
$$= b_0 + \frac{1}{2b_1 + \frac{1}{b_2}}$$
$$= [b_0, 2b_1, b_2]$$

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Case 3 ($n > 2$): We want to show that $[b_0, 2b_1, b_2, \ldots, b_n] = [b_0+1, (-2, 2)^{b_1-1}, -2, b_2+1, b_3, b_4 \ldots, b_n]$. Let $R = [b_3, b_4, \ldots, b_n]$.

$$[b_0 + 1, (-2, 2)^{b_1-1}, -2, b_2 + 1, b_3, b_4 \ldots, b_n]$$

$$= b_0 + 1 + \cfrac{1}{\cfrac{2(b_1-1)(\frac{-2Rb_2-R-2}{Rb_2+R+1})+2(b_1-1)+\frac{-2Rb_2-R-2}{Rb_2+R+1}}{1-2(b_1-1)\frac{-2Rb_2-R-2}{Rb_2+R+1}-2(b_1-1)}} \quad \text{(Apply Identity 1)}$$

$$= b_0 + 1 + \cfrac{Rb_2 - 2b_1 - R - 2Rb_1b_2 + 1}{R + 2b_1 + 2Rb_1b_2}$$

$$= b_0 + \cfrac{1}{\cfrac{R+2b_1+2Rb_1b_2}{Ra_2+1}}$$

$$= b_0 + \cfrac{1}{2b_1 + \cfrac{R}{Rb_2+1}}$$

$$= b_0 + \cfrac{1}{2b_1 + \cfrac{1}{b_2+\frac{1}{R}}}$$

$$= [b_0, 2b_1, b_2, b_3, \ldots, b_n]$$

$\square$

SUBSTITUTION 2. *Let $n \in \mathbb{N}$. Let $b_0 \in \mathbb{Z}$ and $b_1 \in \mathbb{N}_0$. If $n = 2$ then let $b_2 \in \mathbb{Z}\backslash\{0,1\}$. If $n \geq 3$ then let $b_i \in \mathbb{Z}^*$ for all $2 \leq i \leq n$. Then $[b_0, -2b_1, b_2, b_3, \ldots, b_n] = [b_0-1, (2, -2)^{b_1}, b_2-1, 2, b_3, b_4, \ldots, b_n]$. In particular, $[b_0, -2b_1, b_2] = [b_0-1, (2, -2)^{b_1}, 2, b_2-1]$.*

PROOF. This proof will be conducted in three parts, analogous to the proof of Substitution 1.

Case 1 ($n = 1$): We want to show that $[b_0, -2b_1] = [b_0 - 1, (2, -2)^{b_1-1}, 2]$.

$$[b_0 - 1, (2, -2)^{b_1-1}, 2]$$

$$= b_0 - 1 + \cfrac{1}{\frac{2(b_1-1)(2)-2(b_1-1)+(2)}{2(b_1-1)(2)-2(b_1-1)+1}} \quad \text{(Apply Identity 1)}$$

$$= b_0 - 1 + \cfrac{2b_1 - 1}{2b_1}$$

$$= b_0 + \cfrac{1}{-2b_1}$$

$$= [b_0, -2b_1]$$

Case 2 ($n = 2$): We want to show that $[b_0, -2b_1, b_2] = [b_0-1, (2, -2)^{b_1-1}, 2, b_2-1]$.

$$[b_0 - 1, (2, -2)^{b_1-1}, 2, b_2 - 1]$$

$$= b_0 - 1 + \cfrac{1}{\cfrac{2(b_1-1)(\frac{2b_2-1}{b_2-1})-2(b_1-1)+\frac{2b_2-1}{b_2-1}}{2(b_1-1)\frac{2b_2-1}{b_2-1}-2(b_1-1)+1}} \quad \text{(Apply Identity 1)}$$

$$= b_0 - 1 + \cfrac{2b_1b_2 - b_2 - 1}{2b_1b_2 - 1}$$

$$= b_0 + \cfrac{1}{\frac{2b_1b_2-1}{-b_2}}$$

$$= b_0 + \cfrac{1}{-2b_1 + \frac{1}{b_2}}$$

$$= [b_0, -2b_1, b_2]$$

Case 3 ($n > 2$): We want to show that $[b_0, -2b_1, b_2, \ldots, b_n] = [b_0-1, (2, -2)^{b_1-1}, 2, b_2-1, b_3, b_4 \ldots, b_n]$. Let $R = [b_3, b_4 \ldots, b_n]$.

$$[b_0 - 1, (2, -2)^{b_1 - 1}, 2, b_2 - 1, b_3, b_4 \ldots, b_n]$$

$$= b_0 - 1 + \cfrac{1}{\cfrac{2(b_1-1)(\frac{2Rb_2+2-R}{Rb_2+1-R})+2(b_1-1)+\frac{2Rb_2+2-R}{Rb_2+1-R}}{1-2(b_1-1)\frac{2Rb_2+2-R}{Rb_2+1-R}-2(b_1-1)}} \quad \text{(Apply Identity 1)}$$

$$= b_0 - 1 + \cfrac{Rb_2 - R - 2b_2 - 2Rb_1b_2 - 1}{2b_1 - R + 2Rb_1b_2}$$

$$= b_0 + \cfrac{1}{\frac{R-2b_1-2Rb_1b_2}{Ra_2+1}}$$

$$= b_0 + \cfrac{1}{-2b_1 + \frac{R}{Rb_2+1}}$$

$$= b_0 + \cfrac{1}{-2b_1 + \cfrac{1}{b_2+\frac{1}{R}}}$$

$$= [b_0, -2b_1, b_2, b_3, \ldots, b_n]$$

$\square$

SUBSTITUTION 3. *Let $n \in \mathbb{N}$. Let $b_0 \in \mathbb{Z}$ and $b_1 \in \mathbb{N}_0$. If $n = 2$ then let $b_2 \in \mathbb{Z} \setminus \{0, -1\}$. If $n \geq 3$ then let $b_i \in \mathbb{Z}^*$ for all $2 \leq i \leq n$. Then $[b_0, 2b_1+1, b_2, b_3, \ldots, b_n] = [b_0 + 1, (-2, 2)^{b_1}, -b_2 - 1, -b_3, -b_4, \ldots, -b_n]$. In particular, $[b_0, 2b_1 + 1, b_2] = [b_0 + 1, (-2, 2)^{b_1}, -b_2 - 1]$.*

PROOF. We will prove this substitution rule in three parts.

Case 1 ($n = 1$): We want to show that $[b_0, 2b_1 + 1] = [b_0 + 1, (-2, 2)^{b_1}]$.

$$[b_0, 2b_1 + 1]$$
$$= [b_0, 2b_1, 1]$$
$$= [b_0 + 1, (-2, 2)^{b_1 - 1}, -2, 2] \quad \text{(Apply Substitution 1)}$$
$$= [b_0 + 1, (-2, 2)^{b_1}]$$

Case 2 ($n = 2$): We want to show that $[b_0, 2b_1+1, b_2] = [b_0+1, (-2, 2)^{b_1}, -b_2 - 1]$.

$$[b_0 + 1, (-2, 2)^{b_1}, -b_2 - 1]$$
$$= b_0 + 1 + \cfrac{1}{\frac{2b_1(-b_2-1)+2b_1+(-b_2-1)}{1-2b_1(-b_2-1)-2b_1}} \quad \text{(Apply Identity 2)}$$
$$= b_0 + 1 + \cfrac{-2b_1b_2 - 1}{2b_1b_2 + b_2 + 1}$$
$$= b_0 + \cfrac{1}{\frac{2b_1b_2+b_2+1}{b_2}}$$
$$= b_0 + \cfrac{1}{2b_1 + 1 + \frac{1}{b_2}}$$
$$= [b_0, 2b_1 + 1, b_2]$$

Case 3 ($n > 2$): We want to show that $[b_0, 2b_1 + 1, b_2, \ldots, b_n] = [b_0 + 1, (-2, 2)^{b_1}, -b_2 - 1, -b_3, -b_4 \ldots, -b_n]$. Let $R = [b_3, b_4, \ldots, b_n]$. Note $-R = [-b_3, -b_4, \ldots, -b_n]$.

$$[b_0 + 1, (-2, 2)^{b_1}, -b_2 - 1, -b_3, -b_4 \ldots, -b_n]$$

$$= \quad b_0 + 1 + \cfrac{1}{\cfrac{2b_1(\frac{Rb_2+R+1}{-R})+2b_1+\frac{Rb_2+R+1}{-R}}{1-2b_1(\frac{Rb_2+R+1}{-R})-2b_1}} \qquad \text{(Apply Identity 2)}$$

$$= \quad b_0 + 1 + \cfrac{-R - 2b_1^2 - 2Rb_1b_2}{2b_1 + R + Rb_2 + 2Rb_1b_2 + 1}$$

$$= \quad b_0 + \cfrac{1}{\frac{2b_1+R+Rb_2+2Rb_1b_2+1}{1+Rb_2}}$$

$$= \quad b_0 + \cfrac{1}{2b_1 + 1 + \frac{R}{Rb_2+1}}$$

$$= \quad b_0 + \cfrac{1}{2b_1 + 1 + \cfrac{1}{b_2 + \frac{1}{R}}}$$

$$= \quad [b_0, 2b_1 + 1, b_2, b_3, \ldots, b_n]$$

$\square$

SUBSTITUTION 4. *Let $n \in \mathbb{N}$. Let $b_0 \in \mathbb{Z}$ and $b_1 \in \mathbb{N}_0$. If $n = 2$ then let $b_2 \in \mathbb{Z} \setminus \{0, 1\}$. If $n \geq 3$ then let $b_i \in \mathbb{Z}^*$ for all $2 \leq i \leq n$. Then $[b_0, -2b_1-1, b_2, b_3, \ldots, b_n] = [b_0 - 1, (2, -2)^{b_1}, -b_2 + 1, -b_3, -b_4, \ldots, -b_n]$. In particular, $[b_0, 2b_1 + 1, b_2] = [b_0 + 1, (-2, 2)^{b_1}, -b_2 - 1]$.*

PROOF. We will prove this substitution rule in three parts.

Case 1 ($n = 1$): We want to show that $[b_0, -2b_1 - 1] = [b_0 - 1, (2, -2)^{b_1}]$.

$$[b_0, -2b_1 - 1]$$
$$= \quad [b_0, -2b_1, -1]$$
$$= \quad [b_0 - 1, (2, -2)^{b_1-1}, 2, -2] \quad \text{(Apply Substitution 2)}$$
$$= \quad [b_0 - 1, (2, -2)^{b_1}]$$

Case 2 ($n = 2$): We want to show that $[b_0, -2b_1-1, b_2] = [b_0-1, (2, -2)^{b_1}, -b_2+1]$.

$$[b_0 - 1, (2, -2)^{b_1}, -b_2 + 1]$$

$$= \quad b_0 - 1 + \cfrac{1}{\frac{2b_1(-b_2+1)-2b_1+(-b_2+1)}{1+2b_1(-b_2+1)-2b_1}} \quad \text{(Apply Identity 2)}$$

$$= \quad b_0 - 1 + \cfrac{2b_1b_2 - 1}{2b_1b_2 + b_2 - 1}$$

$$= \quad b_0 + \cfrac{1}{\frac{-2b_1b_2-b_2+1}{b_2}}$$

$$= \quad b_0 + \cfrac{1}{-2b_1 - 1 + \frac{1}{b_2}}$$

$$= \quad [b_0, -2b_1 - 1, b_2]$$

Case 3 ($n > 2$): We want to show that $[b_0, -2b_1 - 1, b_2, \ldots, b_n] = [b_0 - 1, (2, -2)^{b_1}, -b_2 + 1, -b_3, -b_4 \ldots, -b_n]$. Let $R = [b_3, b_4, \ldots, b_n]$. Note $-R = [-b_3, -b_4, \ldots, -b_n]$.

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$$[b_0 - 1, (2, -2)^{b_1}, -b_2 + 1, -b_3, -b_4 \ldots, -b_n]$$

$$= \quad b_0 - 1 + \cfrac{1}{\cfrac{2b_1(\frac{-Rb_2 + R - 1}{R}) - 2b_1 + \frac{-Rb_2 + R - 1}{R}}{1 + 2b_1(\frac{-Rb_2 + R - 1}{R}) - 2b_1}} \qquad \text{(Apply Identity 2)}$$

$$= \quad b_0 - 1 + \cfrac{-R + 2b_1 + 2Rb_1 b_2}{2b_1 - R + Rb_2 + 2Rb_1 b_2 + 1}$$

$$= \quad b_0 - \cfrac{1}{\frac{-2b_1 + R - Rb_2 - 2Rb_1 b_2 + 1}{1 + Rb_2}}$$

$$= \quad b_0 - \cfrac{1}{-2b_1 - 1 + \frac{R}{Rb_2 + 1}}$$

$$= \quad b_0 - \cfrac{1}{-2b_1 - 1 + \frac{1}{b_2 + \frac{1}{R}}}$$

$$= \quad [b_0, -2b_1 - 1, b_2, b_3, \ldots, b_n]$$

$\square$

2.1. An example of the application of the Substitutions. Let us illustrate how the above results can be used to generate a list of all boundary slope continued fractions starting from the simple continued fraction. As an example, suppose we start with $[0, 2a, 2b + 1, 2c]$, where $a, c \in \mathbb{N}$ and $b \in \mathbb{N}_0$. By applying Substitution 1, we can immediately derive another continued fraction: $[1, (-2, 2)^{a-1}, -2, 2b + 2, 2c]$. We will refer to this as *applying Substitution 1 at position 0* as it is the a_0 term, $2a$, that has been replaced by the sequence $-2, 2, \ldots, -2$.

Applying the same substitution at position 2, we get $[1, (-2, 2)^{a-1}, -2, 2b + 3, (-2, 2)^{c-1}, -2]$. We could continue on this path, but it is easy to see that any further substitutions will result in a ± 1 term. Therefore, we return to the original sequence and use Substitution 3 (at position 1) to obtain $[0, 2a + 1, (-2, 2)^b, -2c - 1]$. Finally, applying Substitution 1 at position 2, we have $[0, 2a, 2b + 2, (-2, 2)^{c-1}, -2]$.

Thus, there are five boundary slope continued fractions that can be derived from the simple continued fraction $[0, 2a, 2b + 1, 2c]$: three obtained by substitutions at positions 0, 1, and 2; one by substitutions at 0 and 2; and the original continued fraction itself (with no substitutions). These are precisely the fractions obtained by applying substitutions at non-adjacent positions.

Note that when a substitution is applied at position i, the element a_i is replaced by $(a_i - 1)$ ± 2's and the adjacent terms a_{i-1} and a_{i+1} both have their magnitude increased by one. We will return to these observations when proving Theorem 1.

3. Proof of Theorem 2

In this section we will prove Theorem 2, that the boundary slope continued fractions are among the fractions obtained by applying substitutions at non-adjacent positions in the original simple continued fraction. Our strategy is to first review Langford's argument [11] that the boundary slopes are determined by the leaves of a binary tree. We then show, by induction, that applying substitutions at non-adjacent positions accounts for all the leaves of the tree.

3.1. The boundary slope binary tree. Before we can prove Theorem 2, we must first state a lemma. The straightforward proof by induction may be found in Langford [11] which is also the source for the following definition: the *kth subexpansion* of $[c; a_0, \ldots, a_n]$ is the continued fraction $[0, a_k, \ldots, a_n]$ where $0 \le k \le n$.

LEMMA 1. *Let $[c; a_0, \ldots, a_n]$ be a boundary slope continued fraction, that is, $|a_i| \geq 2$ $(0 \leq i \leq n)$. Then every subexpansion r of $[c; a_0, \ldots, a_n]$ satisfies $|r| < 1$.*

As Langford [11] has shown, a complete list of boundary slope continued fractions for $K(p/q)$, where $0 < p/q < 1$, can be calculated by means of a binary tree. We will now outline the creation of this binary tree which follows from Lemma 1.

The root vertex is labeled with the fraction $\frac{p}{q}$ and the two edges coming from the root are labeled $0 = \lfloor \frac{p}{q} \rfloor$ and $1 = \lceil \frac{p}{q} \rceil$. At every other vertex in the tree, we arrive with the first k terms in a continued fraction for $\frac{p}{q}$ and a rational number r representing the $(k-1)$st subexpansion. The k terms are found as labels of the edges of the tree starting from the root and continuing to the vertex in question. We label the vertex with r. Since, by Lemma 1, any kth subexpansion is less than one in absolute value, we know that the next term in the continued fraction, a_{k-1}, is within 1 of $1/r$: $|a_{k-1} - 1/r| < 1$. However, a_{k-1} is an integer. Therefore, a_{k-1} is either the floor $\lfloor 1/r \rfloor$ or the ceiling $\lceil 1/r \rceil$ of $1/r$. If $1/r$ is not an integer, there will be two edges coming out of the vertex, one labeled with $\lfloor 1/r \rfloor$, and the other labeled with $\lceil 1/r \rceil$. Since $|r| < 1$, neither of these arrows is 0. If either is ± 1, we terminate that edge with a leaf labeled "$\cancel{\exists}$" to indicate that this path does not lead to a boundary slope continued fraction. (When we refer to the leaves of the binary tree below, we will be excluding these "dead" leaves.) If $1/r$ is an integer, then, there is only one edge coming out of the vertex. Label the edge with $1/r$ and label the leaf vertex at the end of this edge with the continued fraction expansion for $\frac{p}{q}$ given by the labels of the edges from the root to the leaf.

For example, Figure 2.1 shows the binary tree for the fraction 2/7 (which corresponds to the 5_2 knot).

Thus, by Lemma 1, the algorithm used to construct the tree will provide all the boundary slope continued fractions of $\frac{p}{q}$ as leaf vertices.

3.2. Binary tree from substitutions. Now, let's prove the theorem by showing that the leaves of Langford's binary tree (and therefore the set of boundary slopes) correspond to applying substitutions at non-adjacent positions in the simple continued fraction.

THEOREM 2. *The boundary slope continued fractions of $K(p/q)$ are among the continued fractions obtained by applying substitutions at non-adjacent positions in the simple continued fraction of p/q.*

PROOF. We proceed by induction on the length n of the simple continued fraction $[0, a_0, a_1, \ldots, a_n]$.

Case 1 ($n = 0$): Here, $p/q = 1/a_0$. We wish to show that the boundary slope continued fractions are among the two continued fractions given by substituting or not at position 0. There are three subcases. (To simplify the exposition, we will not be considering the, very similar, trees that arise when the terms a_i are negative although they may be required as part of our induction.)

<u>Subcase 1</u> ($a_0 = 1$): In this case, the tree is shown in Figure 2.2. There are no boundary slope continued fractions in this case. (Actually, here $\frac{p}{q} = 1$, so we've violated our assumption that $\frac{p}{q} < 1$. Ordinarily, we would represent this knot, the unknot, by [0] and that would also be the only boundary slope. We include this case as it may arise as part of our induction.) Thus, it is true that all boundary

$$\frac{2}{7}$$

$0 = \lfloor \frac{2}{7} \rfloor$ $1 = \lceil \frac{2}{7} \rceil$

$(\frac{2}{7} - 0)$ $(\frac{2}{7} - 1) = -\frac{5}{7}$

$3 = \lfloor \frac{7}{2} \rfloor$ $4 = \lceil \frac{7}{2} \rceil$ $-1 = \lceil -\frac{7}{5} \rceil$ $-2 = \lfloor -\frac{7}{5} \rfloor$

$(\frac{7}{2} - 3) = \frac{1}{2}$ $(\frac{7}{2} - 4) = -\frac{1}{2}$ $\nexists$ $(-\frac{7}{5} + 2) = \frac{3}{5}$

$2 = \lceil \frac{5}{3} \rceil$

2 -2 $1 = \lfloor \frac{5}{3} \rfloor$

$(\frac{5}{3} - 2) = -\frac{1}{3}$

$[0, 3, 2]$ $[0, 4, -2]$ -3 $\nexists$

$[1, -2, 2, -3]$
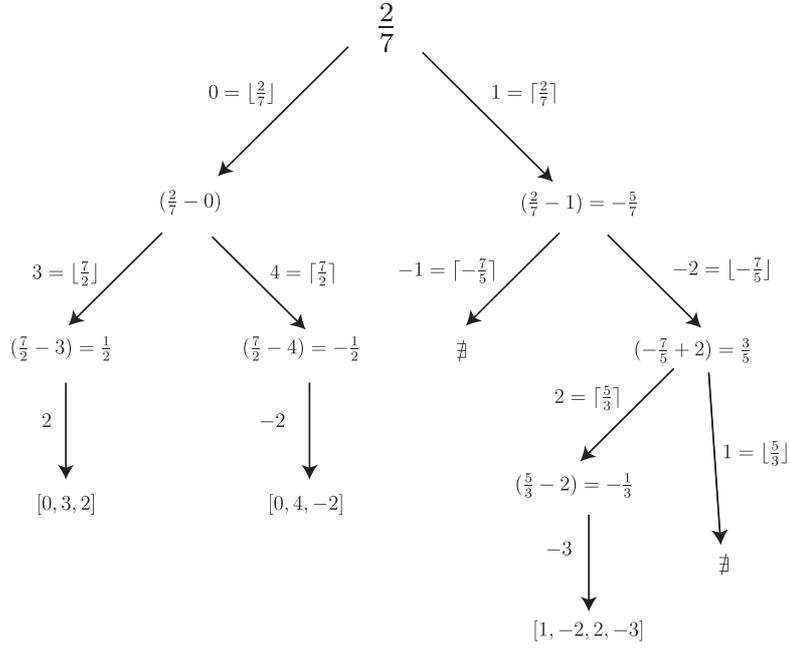
FIGURE 2.1. The boundary slope binary tree for $\frac{p}{q} = \frac{2}{7}$ (the 5_2 knot).

$$[0, 1] = 1$$

$1 = \lfloor 1 \rfloor = \lceil 1 \rceil$

$\nexists$
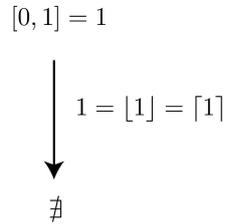
FIGURE 2.2. The binary tree for $[0, 1]$.

slope continued fractions are among the two continued fractions $[0, 1]$ and $[1]$ given by substituting or not at position 0.

<u>Subcase 2</u> ($a_0 = 2a$, $a \geq 1$): The binary tree is shown in Figure 2.3. There are two boundary slope continued fractions, and they are the fractions $[0, a_0]$ and $[1, (-2, 2)^a, -2]$ given by substituting or not at position 0.

<u>Subcase 3</u> ($a_0 = 2a + 1$, $a \geq 1$): The binary tree is shown in Figure 2.4. The two boundary slope continued fractions $[0, a_0]$ and $[1, (-2, 2)^a]$ are those given by substituting or not at position 0.

Case 2 ($n = 1$): Our goal is to show that the boundary slope continued fractions are among the fractions given by substituting at position 0, at position 1, and by not substituting at all. The result of substitution at position 0 will depend

$$[0, 2a] = 1/2a$$

$$0 = \lfloor 1/2a \rfloor$$

$$1 = \lceil 1/2a \rceil$$

$$(1/2a - 0)$$

$$(1/2a - 1) = -\frac{2a-1}{2a}$$

$$-1 = \lceil \frac{-2a}{2a-1} \rceil$$

$$-2 = \lfloor \frac{-2a}{2a-1} \rfloor$$

$$2a$$

$$\nexists$$

$$[0, 2a]$$

$$(\frac{-2a}{2a-1} + 2) = \frac{2a-2}{2a-1}$$

$$2 = \lceil \frac{2a-1}{2a-2} \rceil$$

$$1 = \lfloor \frac{2a-1}{2a-2} \rfloor$$

$$\nexists$$

$$(\frac{2a-1}{2a-2} - 2) = -\frac{2a-3}{2a-2}$$

$$-1$$

$$-2$$

$$\nexists$$

$$-\frac{2a-(2a-1)}{2a-(2a-2)} = -\frac{1}{2}$$

$$-2$$
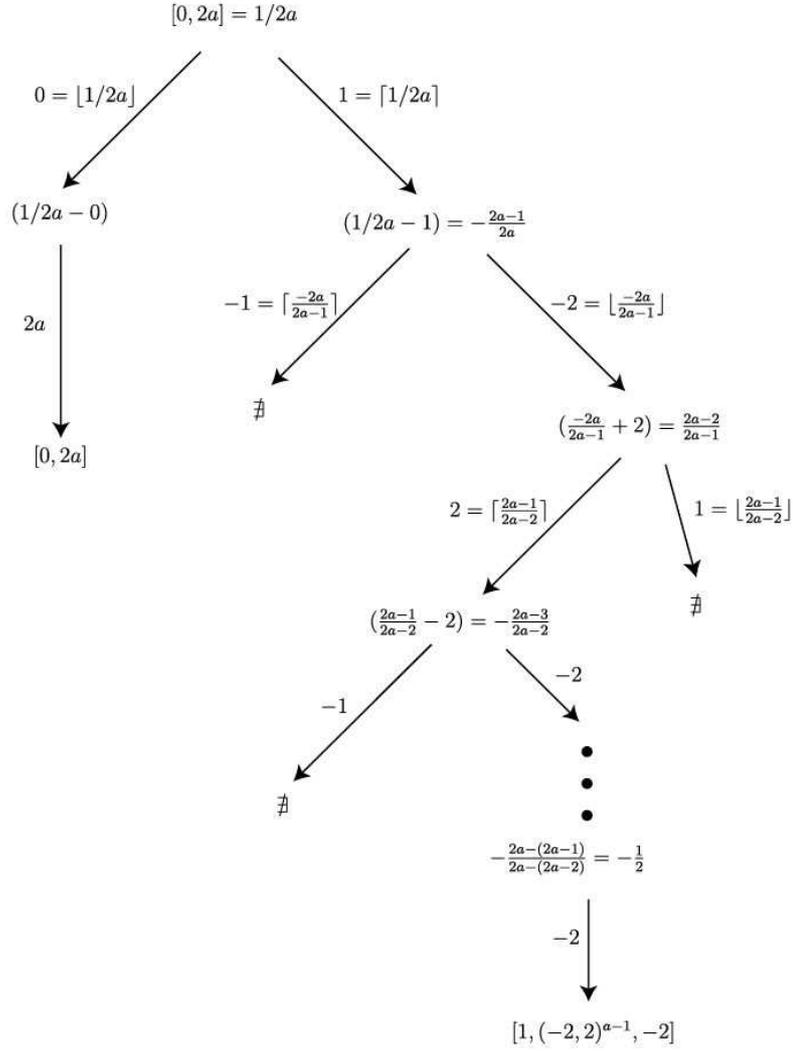
$$[1, (-2, 2)^{a-1}, -2]$$

FIGURE 2.3. The binary tree for $[0, 2a]$.

on whether a_0 is even or odd:

$$[0, 2a, a_1] \xrightarrow{\text{Sub. } 1} [1, (-2, 2)^{(a-1)}, -2, a_1 + 1]$$

$$[0, 2a + 1, a_1] \xrightarrow{\text{Sub. } 3} [1, (-2, 2)^a, -a_1 - 1]$$

Similarly, substitution at position 1 depends on the parity of a_1:

$$[0, a_0, 2b] \xrightarrow{\text{Sub. } 1} [0, a_0 + 1, (-2, 2)^{(b-1)}, -2]$$

$$[0, a_0, 2b + 1] \xrightarrow{\text{Sub. } 3} [0, a_0 + 1, (-2, 2)^b]$$

As Figure 2.5 shows, these two boundary slopes, along with the original contin-

$$[0, 2a+1] = 1/(2a+1)$$

0 1

$1/(2a+1)$

-1 -2

$2a+1$

$\nexists$

$[0, 2a+1]$

2 1

$\nexists$

-1 -2

$\nexists$

$\bullet$
$\bullet$
$\bullet$

2 1

$\nexists$

-1 -2

$\nexists$ $1/2$

2

$$[1, (-2, 2)^a]$$
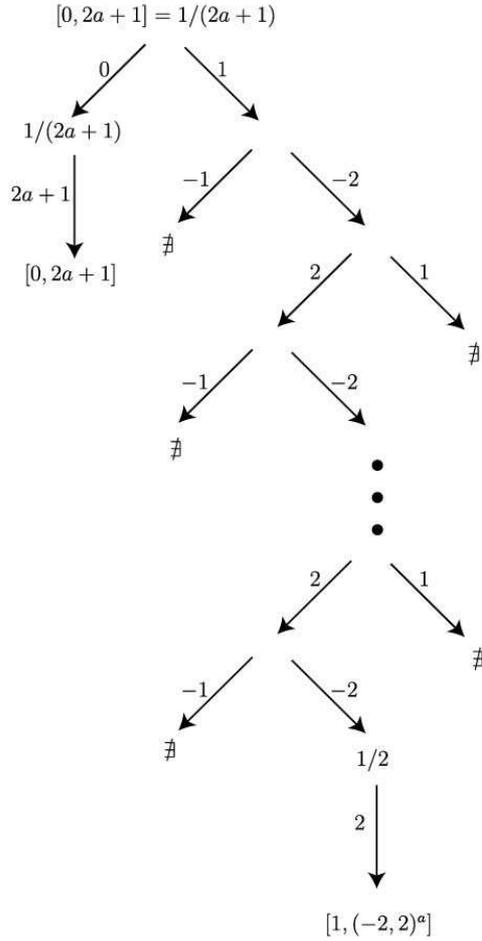
FIGURE 2.4. The binary tree for $[0, 2a+1]$.

ued fraction $[0, a_0, a_1]$ (no substitutions) are precisely those that arise in the binary tree. Note that if, for example, a_0 or a_1 is 1, then the $[0, a_0, a_1]$ leaf is not in fact a boundary slope continued fraction. The point is that all leaves of the binary tree are included in the set of continued fractions obtained by substitutions at non-adjacent positions. So, every boundary slope continued fraction appears in this set.

Case 3 ($n = 2$): This case will illustrate how the induction works. There are five continued fractions given by substitutions at non-adjacent positions (compare with the example of Section 2.1 of this chapter): three obtained by substitutions at positions 0, 1, and 2; one by substitutions at 0 and 2; and the original continued fraction itself (with no substitutions). Let us denote these choices of substitutions by a sequence of three 0's and 1's where a 1 in the ith place denotes a substitution at that ith position. Thus, the five continued fractions will be denoted 100, 010, 001, 101, and 000.

We can think of the binary tree (Figure 2.6) as being a union of two subtrees.

$$[0, a_0, a_1] = \frac{1}{a_0 + 1/a_1}$$

$$0 = \lfloor \frac{1}{a_0 + 1/a_1} \rfloor \qquad 1 = \lceil \frac{1}{a_0 + 1/a_1} \rceil$$

$$(\frac{1}{a_0 + 1/a_1} - 0) \qquad (\frac{1}{a_0 + 1/a_1} - 1) = -\frac{a_0 a_1 + 1 - a_1}{a_0 a_1 + 1}$$

$$a_0 = \lfloor a_0 + 1/a_1 \rfloor \qquad a_0 + 1 = \lceil a_0 + 1/a_1 \rceil$$

$$-2 = \lfloor -1 - \frac{a_1}{a_0 a_1 + 1 - a_1} \rfloor = \lfloor -\frac{a_0 a_1 + 1}{a_0 a_1 + 1 - a_1} \rfloor$$

$$-1 = \lceil -\frac{a_0 a_1 + 1}{a_0 a_1 + 1 - a_1} \rceil$$

$$(a_0 + 1/a_1 - a_0) = 1/a_1$$

$$\nexists \qquad (-\frac{a_0 a_1 + 1}{a_0 a_1 + 1 - a_1} + 2) = \frac{a_0 a_1 + 1 - 2a_1}{a_0 a_1 + 1 - a_1}$$

$$a_1 \qquad (a_0 + 1/a_1 - (a_0 + 1)) = -\frac{a_1 - 1}{a_1}$$

$$2 \qquad 1 = \lfloor 1 + \frac{a_1}{a_0 a_1 + 1 - 2a_1} \rfloor = \lfloor \frac{a_0 a_1 + 1 - a_1}{a_0 a_1 + 1 - 2a_1} \rfloor$$

$$-1 = \lceil -\frac{a_1}{a_1 - 1} \rceil \qquad -2 = \lfloor -\frac{a_1}{a_1 - 1} \rfloor$$

$$\nexists$$

$$(\frac{a_0 a_1 + 1 - a_1}{a_0 a_1 + 1 - 2a_1} - 2) = -\frac{a_0 a_1 + 1 - 3a_1}{a_0 a_1 + 1 - 2a_1}$$

$$[0, a_0, a_1]$$

$$\nexists$$

$$-1 \qquad -2 = \lfloor -1 - \frac{a_1}{a_0 a_1 + 1 - 3a_1} \rfloor = \lfloor -\frac{a_0 a_1 + 1 - 2a_1}{a_0 a_1 + 1 - 3a_1} \rfloor$$

$$2 \qquad 1$$

$$\nexists$$

$$\bullet$$
$$\bullet$$
$$\bullet$$

$$\nexists$$

$$\bullet$$
$$\bullet$$
$$\bullet$$

$$\pm 1/2$$

$$\pm 2$$

$$\pm 1 \qquad \pm 2$$

$$\nexists \qquad \pm \frac{a_0 a_1 + 1 - a_0 a_1}{a_0 a_1 + 1 - (a_0 - 1)a_1} = \pm 1/(1 + a_1)$$

$$[0, a_0 + 1, -2, 2, -2, 2, \ldots, \pm 2]$$

$$\pm (a_1 + 1)$$

$$[1, -2, 2, -2, 2, \ldots, \pm 2, \pm (a_1 + 1)]$$
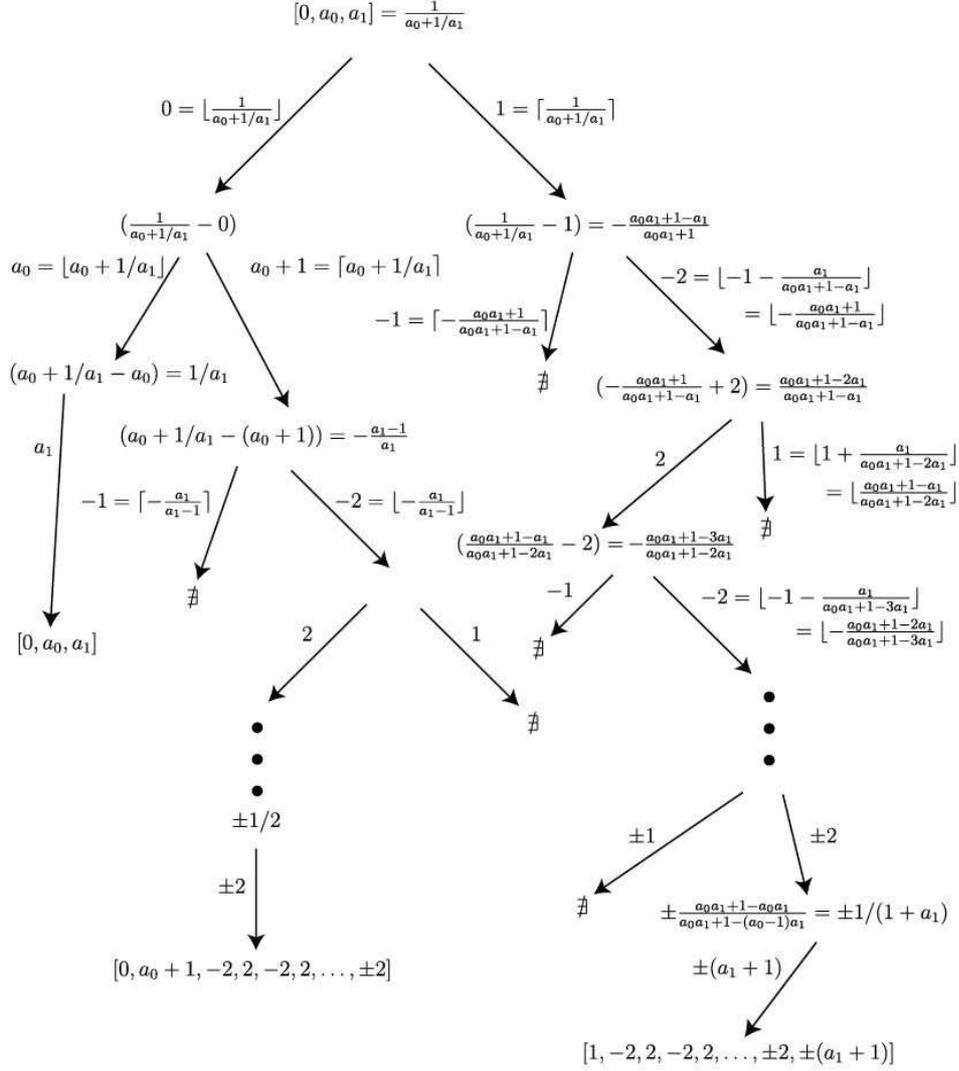
FIGURE 2.5. The binary tree for $[0, a_0, a_1]$.

The one at left corresponds to making no substitution at position 0. This subtree ends in the three boundary slopes which have: no substitutions (000); substitution at position 1 (010); and substitution at position 2 (001), i.e., the sequences that begin in 0. This subtree is essentially the same as that for the $[0, a_1, a_2]$ continued fraction (compare Figure 2.5) as we can obtain these three sequences by adding a 0 at the front of the three boundary slopes sequences 00, 10, and 01 of that case. The other subtree corresponds to making a substitution at position 0 and no substitution at position 1. This subtree contains the remaining two boundary slopes: substitution at position 0 (100); and substitution at positions 0 and 2 (101), i.e., sequences that begin in 10. This subtree is similar to that for $[0, a_2]$ (compare
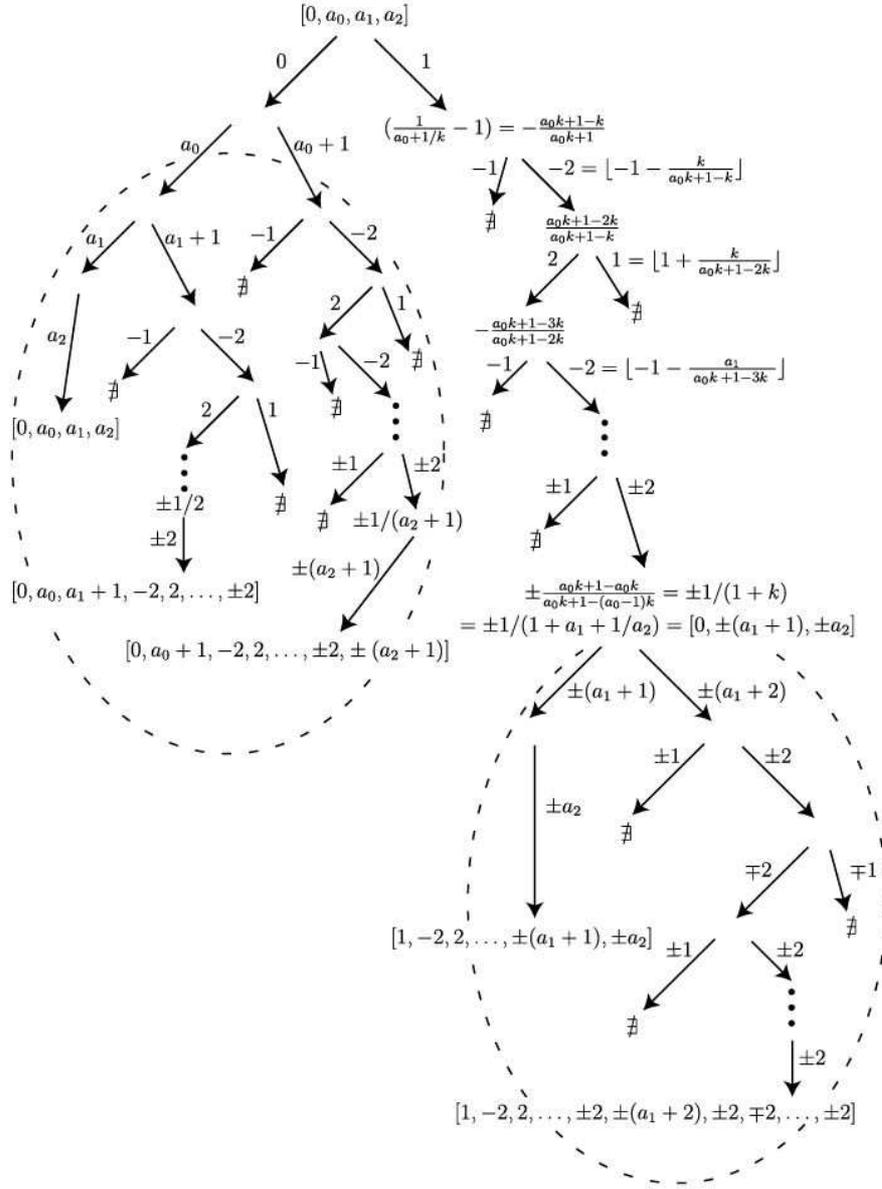
$$[0, a_0, a_1, a_2]$$

FIGURE 2.6. The $[0, a_0, a_1, a_2]$ tree is a union of two subtrees.

Figure 2.3) as it remains only to decide whether or not to substitute in the second position. Again, some of these five sequences may not result in a boundary slope continued fraction, for example, if one of the a_i is 1. However, every leaf of the tree will be included in the set of continued fractions obtained by substituting at non-adjacent positions.

Case 4 ($n \geq 3$): As in Case 3, we can decompose the binary tree (Figure 2.7) into two subtrees. One corresponds to sequences that begin with 0, the other to
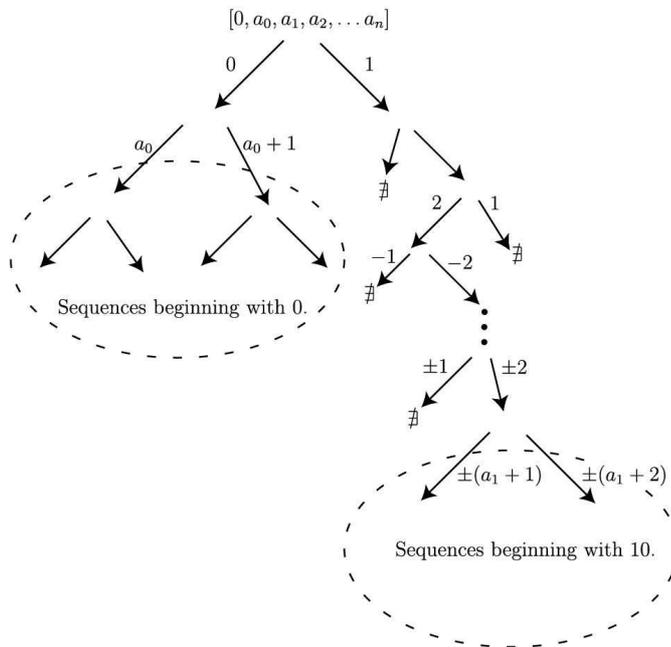
FIGURE 2.7. The general case also results in two subtrees.

sequences beginning with 10. The first will be, essentially, the tree that arises from the simple continued fraction $[0, a_1, a_2, \ldots, a_n]$. By induction, the leaves of this subtree correspond to non-adjacent substitutions in this simple continued fraction. By its placement in the $[0, a_0, a_1, \ldots, a_n]$ tree, this ensures that the leaves of this part of the tree will correspond to continued fractions obtained by substitution sequences into $[0, a_0, a_1, \ldots, a_n]$ that begin with 0.

The other subtree is isomorphic to the tree that arises from the simple continued fraction $[0, a_2, a_3, \ldots, a_n]$. By induction, the leaves of the subtree correspond to substitutions into this continued fraction. By its placement in the tree for $[0, a_0, a_1, \ldots, a_n]$, the leaves here can be obtained by non-adjacent substitutions into that continued fraction that begin with 10.

Thus, every leaf of the binary tree and, therefore, every boundary slope continued fraction can be obtained by non-adjacent substitutions into the simple continued fraction.

$\square$

COROLLARY 2. *If $\frac{p}{q} = [0, a_0, a_1, \ldots, a_n]$ is a simple continued fraction, then $K(p/q)$ has at most F_{n+2} boundary slopes where F_n is the nth Fibonacci number.*

PROOF. This result has been previously proven in [**5**]. We will provide an alternative proof, based on our substituion rules. We have shown that the boundary

slope continued fractions lie among those given by substitution at non-adjacent positions which in turn are in bijection with sequences of $n+1$ 0's or 1's containing no pair of consecutive 1's. Thus the number of boundary slopes is at most P_n, where P_n is the number of 0, 1 sequences of length $n+1$ with no consecutive 1's. We will show that $P_n = F_{n+2}$ by induction.

There are two base cases. If $n = 0$, there are two sequences: 0 and 1. So, $P_0 = 2 = F_2$. For $n = 1$, there are three sequences: 00, 10, and 01. So, $P_1 = 3 = F_3$.

For the inductive step, sequences of length $n+1$ are obtained by either adding a 0 to the beginning of a n sequence or 10 to the beginning of a $n-1$ sequence. Thus $P_n = P_{n-1} + P_{n-2} = F_{n+1} + F_n = F_{n+2}$. $\square$

In general, F_{n+2} is an overestimate since the continued fractions obtained by substitutions will not necessarily have terms at least two in absolute value. In particular, if the simple continued fraction includes any 1's, then the continued fraction obtained by making no substitutions $(000\ldots 0)$ will not be a boundary slope continued fraction. Moreover, different boundary slope continued fractions could result in the same boundary slope. For example, this will occur when, in the simple continued fraction, we have two equal terms separated by an even distance: $a_i = a_{i+2k}$.

4. Proof of Theorem 1

In this section we prove Theorem 1. We will argue that the maximum and minimum boundary slopes are given by the substitution patterns $010101\cdots$ and $101010\cdots$ respectively. This will allow us to compare the diameter to the crossing number.

Denote by $\partial[s_0 s_1 \cdots s_n]$ the boundary slope obtained by applying the substitution pattern $s_0 s_1 \cdots s_n$ to some simple continued fraction $[c; a_0, a_1, \ldots, a_n]$. That is, $s_0, s_1, \ldots, s_n$ is a sequence of 0's and 1's with no adjacent 1's. Let $\delta[s_0 s_1 \cdots s_n]$ be the $n^+ - n^-$ portion of this boundary slope. Clearly, if S and S' are substitution patterns, then $\partial[S] < \partial[S'] \iff \delta[S] < \delta[S']$ and $\partial[S] = \partial[S'] \iff \delta[S] = \delta[S']$.

A key observation is that, substitution at an "even" position will decrease the value of the boundary slope. This is because, no matter which substitution is made at position $2i$, the positive number a_{2i} (which counted towards n^+) will be replaced by a sequence of $(a_{2i} - 1)$ ± 2's that count towards n^-. Similarly, substituting at an "odd" position will increase the value:

Lemma 2.

(1) $\partial[1] < \partial[0]$

(2) $\partial[10 s_2 s_3 \cdots s_n] < \partial[00 s_2 s_3 \cdots s_n] < \partial[01 s_2 s_3 \cdots s_n]$

(3) $\partial[001 s_3 s_4 \cdots s_n] < \partial[000 s_3 s_4 \cdots s_n] < \partial[010 s_3 s_4 \cdots s_n]$

Moreover,

$\partial[t_1 \cdots t_{2n} 10 s_1 \cdots s_n] < \partial[t_1 \cdots t_{2n} 00 s_1 \cdots s_n] < \partial[t_1 \cdots t_{2n} 01 s_1 \cdots s_n]$ *and*

$\partial[t_1 \cdots t_{2n+1} 01 s_1 \cdots s_n] < \partial[t_1 \cdots t_{2n+1} 00 s_1 \cdots s_n] < \partial[t_1 \cdots t_{2n+1} 10 s_1 \cdots s_n]$

PROOF. Equation 1: In this case, $\frac{p}{q} = [0, a_0]$ and we are comparing the boundary slope $\partial[1]$ obtained by a substitution at position 0 with that $\partial[0]$ obtained by no substitutions.

Let $\delta[0] = n^+ - n^-$. Then

$$
\begin{aligned}
\delta[1] &= (n^+ - 1) - (n^- + a_0 - 1) \\
&= n^+ - n^- - a_0 \\
&< n^+ - n^- \\
&= \delta[0].
\end{aligned}
$$

Equation 2: Let $\delta[00s_2s_3 \cdots s_n] = n^+ - n^-$. Notice that $n = 1$ implies a length two continued fraction. Then

$$
\begin{aligned}
\delta[10s_2s_3 \cdots s_n] &= n^+ - n^- - a_0 \\
&< n^+ - n^- = \delta[00s_2s_3 \cdots s_n] \\
&< n^+ - n^- + a_1 \\
&= (n^+ + a_1 - 1) - (n^- - 1) \\
&= \delta[01s_2s_3 \cdots s_n].
\end{aligned}
$$

Equation 3: Let $\delta[000s_3s_4 \cdots s_n] = n^+ - n^-$. Notice that $n = 2$ implies a length three continued fraction. Then

$$
\begin{aligned}
\delta[001s_3s_4 \cdots s_n] &= n^+ - n^- - a_2 \\
&< n^+ - n^- = \delta[000s_1s_2 \cdots s_n] \\
&< n^+ - n^- + a_1 \\
&= (n^+ + a_1 - 1) - (n^- - 1) \\
&= \delta[010s_3s_4 \cdots s_n].
\end{aligned}
$$

The remaining two equations follow since adding the same sequence of substitutions at the beginning of the continued fraction will have a similar effect on all three of the boundary slopes. $\qquad\square$

Let $\frac{p}{q} = [0, a_0, \ldots, a_n]$ be the simple continued fraction for the knot $K = K(p/q)$ where $0 < p/q < 1$. It follows from the lemma that the minimum boundary slope is $\partial[101010 \cdots]$ while the maximum is $\partial[010101 \cdots]$.

Note that these two are indeed boundary slopes; that is, each term in the resulting continued fraction is at least two in absolute value. For example, under the substitution $101010 \cdots$ the even position terms a_{2i} of the original simple continued fraction will be replaced by a sequence of $(a_{2i} - 1)$ ± 2's while the terms in the odd positions will be augmented in absolute value by at least one. Moreover, this substitution pattern will result in a continued fraction for which all terms satisfy the pattern $[-+-+\cdots]$. So, if we let n_1^+ and n_1^- be the numbers used in calculating this boundary slope, we have $n_1^+ = 0$ while n_1^- simply counts the number of terms in the resulting continued fraction. Again, as each a_{2i+1} is replaced by $(a_{2i+1} - 1)$ terms and there are $\lceil n/2 \rceil$ terms resulting from the a_{2i}'s, we have

$$n_1^- = \left\lceil \frac{n}{2} \right\rceil + \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (a_{2i} - 1)$$

$$= \left\lceil \frac{n}{2} \right\rceil - \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) + \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a_{2i}$$

$$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a_{2i} \qquad \text{if } n \text{ is odd}$$

$$= -1 + \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a_{2i} \qquad \text{if } n \text{ is even}$$

Similarly, for $\partial[101010\cdots]$, $n_2^- = 0$ and

$$n_2^+ = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} a_{2i+1} \qquad \text{if } n \text{ is odd}$$

$$= 1 + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} a_{2i+1} \qquad \text{if } n \text{ is even}$$

We can now prove that twice the crossing number of a 2–bridge knot K is equal to the diameter of the boundary slopes.

THEOREM 1. *For K a 2–bridge knot, $D(K) = 2c(K)$.*

PROOF. Let K be a 2–bridge knot with associated fraction p/q. We may assume $0 \leq p/q < 1$. If $p/q = 0$, then K is the unknot and the theorem is valid in this case. So, we will assume $0 < p/q < 1$.

If $[0, a_0, \ldots, a_n] = \frac{p}{q}$ is the simple continued fraction for K, then $c(K) = \sum_{i=0}^{n} a_i$ (see [4]).

The diameter of $B(K)$ is also easy to calculate. If we use the n_1^- and n_2^+ found above, we get $D(K) = 2n_2^+ - 2(n_0^+ - n_0^-) - \left(-2n_1^- - 2(n_0^+ - n_0^-) \right) = 2n_2^+ + 2n_1^-$. At this point, n_1^- and n_2^+ may vary depending on whether n is even or odd. However, the differences cancel each other out in either instance, leaving us with

$$D(K) = 2 \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} a_{2i+1} + 2 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a_{2i}$$

$$= 2 \sum_{i=0}^{n} a_i$$

This concludes the proof that $2c(K) = D(K)$. $\qquad\square$

5. Knots with at most four boundary slopes

In this section we characterize 2–bridge knots with four or fewer boundary slopes. We will prove Theorem 3 in two steps by first examining knots with at most three boundary slopes and then those with four boundary slopes.

THEOREM 4. *Let $K = K(p/q)$ be a 2–bridge knot. If K has only two distinct boundary slopes, then K is a torus knot. If K has precisely three boundary slopes, then $p|(q \pm 1)$ or $(q - p)|(q \pm 1)$.*

We will break the proof up into several lemmas which taken together imply the theorem.

In the following, let $K = K(p/q)$ be a 2–bridge knot where $0 < p/q < 1$, p and q are relatively prime, and p/q has simple continued fraction $[0, a_0, a_1, \ldots, a_n]$ with $a_n > 1$. We also assume that q is odd (otherwise p/q represents a 2-bridge link and not a knot); although this places constraints on the parity of the a_i, we will not mention these constraints explicitly. Unless otherwise stated, "K has n distinct boundary slopes" should be taken to mean "K has precisely n distinct boundary slopes".

LEMMA 3. *K has two distinct slopes if and only if K is a torus knot.*

PROOF. We proceed with several cases depending on n, the length of the simple continued fraction of p/q. Note that a 2–bridge torus knot will have fraction p/q of the form $1/q$ or $(q - 1)/q$.

Case 1 ($n = 0$): $p/q = [0, a_0] = 1/a_0$. So, K is a torus knot and, by Lemma 2, has two distinct boundary slopes $\partial[0]$ and $\partial[1]$.

Case 2 ($n = 1$): $p/q = [0, a_0, a_1] = a_1/(a_0 a_1 + 1)$. This represents a torus knot only when $a_0 = 1$ (since $a_1 > 1$, by assumption). When $a_0 = 1$, we get boundary slopes $\partial[01]$, $\partial[10]$, and $\partial[00]$ is not a boundary slope (since $|a_0| < 2$). Also, this is a torus knot, since $a_1/(a_1 + 1)$ is of the form $(q - 1)/q$.

Case 3 ($n \geq 2$): As there are at least 3 terms, p/q is not of the form $1/q$ or $(q-1)/q$ and this is not a torus knot. Also, there are at least three distinct boundary slopes (by Lemma 2): $\partial[1010101 \cdots] < \partial[10010101 \cdots] < \partial[01010101 \cdots]$. ☐

LEMMA 4. *If $n = 1$ then $p|(q - 1)$.*

PROOF. $p/q = [0, a_0, a_1] = a_1/(a_0 a_1 + 1)$. Note that $a_1|a_0 a_1$ and $a_0 a_1 = q - 1$. ☐

LEMMA 5. *If $n = 2$, then K has three distinct boundary slopes if and only if either $a_0 = 1$, or else $a_1 = 1$ and $a_0 = a_2$.*

PROOF. By Theorem 2, there are at most five boundary slopes: $\partial[000]$, $\partial[001]$, $\partial[010]$, $\partial[100]$, and $\partial[101]$. Recall that $\partial[S]$ is a boundary slope only if the substitution pattern S results in a continued fraction with each term at least two in absolute value.

($\Rightarrow$) Assume that $a_0 > 1$ and that either $a_1 > 1$ or $a_0 \neq a_2$. By Lemma 2, we have at least three distinct boundary slopes: $\partial[101]$, $\partial[100]$, and $\partial[010]$. We will show that a fourth boundary slope also exists. Case 1 ($a_0 \neq a_2$): In this case, $\partial[001]$ will be a boundary slope different from $\partial[100]$. Indeed, using the proof of Lemma 2, $\partial[001] = \partial[000] - a_2$ while $\partial[100] = \partial[000] - a_0$. $\partial[001]$ is also different from $\partial[101]$ and $\partial[010]$ (by Lemma 2).

Case 2 ($a_0 = a_2$ and $a_1 > 1$): Since $a_1, a_2, a_3 > 1$, $\partial[000]$ is a boundary slope. Further, by Lemma 2, it is different from $\partial[100]$, and it is also different from $\partial[101]$ and $\partial[010]$.

($\Leftarrow$) Case 1 ($a_0 = 1$): $\partial[000]$ and $\partial[001]$ are not boundary slopes, so K has three distinct boundary slopes.

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Case 2 ($a_1 = 1$ and $a_0 = a_2$): $\partial[000]$, once again, is not a boundary slope. Also, $\partial[100] = \partial[001]$ since $a_0 = a_2$. So K has three distinct boundary slopes. $\square$

LEMMA 6. *If $n = 2$ and K has three distinct boundary slopes, then $(q-p)|(q-1)$ or $p|(q+1)$.*

PROOF. Since K has precisely three boundary slopes, Lemma 5 tells us that either $a_0 = 1$, or else $a_1 = 1$ and $a_0 = a_2$.

Case 1 ($a_0 = 1$): $[0, 1, a_1, a_2] = \frac{a_1 a_2 + 1}{a_1 a_2 + a_2 + 1}$. Then $q - p = a_2$, so $(q-p)|(q-1)$.

Case 2 ($a_1 = 1$ and $a_0 = a_2$): $[0, a_0, 1, a_0] = \frac{a_0 + 1}{a_0^2 + 2a_0} = \frac{a_0 + 1}{(a_0 + 1)^2 - 1}$, so, in this case, $p|(q+1)$. $\square$

LEMMA 7. *If $n = 3$ and K has three distinct boundary slopes, then $(q-p)|(q+1)$.*

PROOF. First, we determine the form of the simple continued fraction given that there are precisely three boundary slopes. Note that, by Lemma 2, there exist at least four boundary slope continued fractions, obtained from substitution patterns $0101, 0100, 1001, 1010$. Also note that $\partial[1000]$ must not be a boundary slope, since, if it were, it would be different from $\partial[1010]$, $\partial[0101]$, and $\partial[1001]$, giving us a fourth boundary slope. Similarly, $\partial[0010]$ cannot exist since it would be different from $\partial[1010]$, $\partial[0101]$, and $\partial[0100]$, also giving us a fourth boundary slope. Therefore, since $\partial[1000]$ isn't a boundary slope, $a_2 = 1$. Similarly, since $\partial[0010]$ isn't a boundary slope, $a_0 = 1$. From this, we can also conclude that $\partial[0000]$ and $\partial[0001]$ are not boundary slopes.

Now, we have four boundary slopes: $\partial[1010]$, $\partial[0101]$, $\partial[1001]$, and $\partial[0100]$. In order to have only three distinct boundary slopes, we need two of these to be equal. By Lemma 2, the only possibility is, $\partial[1001] = \partial[0100]$. Using the proof of Lemma 2, we have $\partial[0100] = \partial[0000] + a_1$ and $\partial[1001] = \partial[0000] + a_3 - a_0$. Thus, $a_3 = a_1 + a_0$ and, since $a_0 = 1$, we have $a_3 = a_1 + 1$.

So, the simple continued fraction must be of the form $[0, 1, a, 1, a + 1] = (a^2 + 3a + 1)/(a^2 + 4a + 3) = (a^2 + 3a + 1)/((a + 2)^2 - 1)$. Then, $q - p = a + 2$, and so $(q-p)|(q+1)$. $\square$

LEMMA 8. *If $n \geq 4$, then K has at least four distinct boundary slopes.*

PROOF. When $n = 4$, by Lemma 2, the four boundary slopes $\partial[10101]$, $\partial[01010]$, $\partial[10010]$, and $\partial[10100]$ are all distinct. If $n > 4$, by appending $101010\cdots$ or $010101\cdots$ to the patterns for $n = 4$, we will have still have at least four distinct boundary slopes. $\square$

Theorem 4 is now proved.

Next, we investigate knots with four boundary slopes.

THEOREM 5. *Let $K = K(p/q)$ be a 2–bridge knot. If K has precisely four boundary slopes, then one of the following holds: $p|(q+1)$, $(q-p)|(q+1)$, $(p\pm 1)|q$, or $(q-p\pm 1)|q$.*

PROOF. **Case 1** ($n = 2$): If $n = 2$, then we have at most 5 boundary slopes, yielded by substitutions $\partial[000], \partial[001], \partial[010], \partial[100], \partial[101]$. By Lemma 2, $\partial[101] < \partial[001]$ or $\partial[100] < \partial[000] < \partial[010]$, and $\partial[101], \partial[010], \partial[100]$ must yield distinct boundary slopes. Hence, to obtain four boundary slopes, we need either $\partial[100] = \partial[001]$ and $\partial[000]$ to exist, or we need $\partial[000]$ nonexistent and $\partial[100] \neq \partial[001]$ which both exist. The former may only occur when $a_0 = a_2$ and $a_1 > 1$, yielding a

continued fraction of the form $[0; a, b, a]$. The latter may only occur when $a_0 \neq a_2$ and $a_1 = 1$, yielding a continued fraction of the form $[0; a, 1, b]$.

Let $a, b > 1$ both be odd (to ensure p/q has odd denominator, and therefore represents a knot rather than a link). Then $[0, a, b, a] = \frac{ab+1}{a((ab+1)+1)} = p/q$, resulting in a fraction with $(p+1)|q$.

Those of the form $[0; a, 1, b]$ with $a, b > 1, a \neq b$ and a, b not both even yield the continued fraction $p/q = \frac{b+1}{(b+1)(a+1)-1}$ with $p|(q+1)$.

Case 2 ($n = 3$): If $n = 3$, then four substitution patterns always yield boundary slopes: $\partial[1010], \partial[1001], \partial[0100], \partial[0101]$, with $\partial[1010] < \partial[1001], \partial[0100] < \partial[0101]$. Notice that one of a_0, a_1, a_2 must be 1 since $\partial[0000]$ must not lead to a boundary slope (since it would be different from at $\partial[1010], \partial[1000], \partial[0101]$, and $\partial[0100]$, each of which would exist and be mutually distinct).

Subcase 1 Now we shall consider one of the cases where $\partial[0100] \neq \partial[1001]$, then $a_3 - a_0 \neq a_1$ and the remaining patterns must either not yield boundary slopes, or must yield duplicates of the first four boundary slopes. In particular, notice that $\partial[0001]$ either must not yield a boundary slope or must be equal to $\partial[0100]$. If we assume that $\partial[0001] = \partial[0100]$, then $a_0 > 1$, $a_1 = a_3$, and $a_2 = 1$ (since $\partial[0000]$ must not exist). Similarlym $\partial[0010]$ must either not exist ($a_0 > 1$, contradicting a previous assumption) or $\partial[0010] = \partial[1001]$, leading to $a_3 - a_0 = -a_2$, i.e. $a_0 = a_3 + 1$. This yields continued fractions of the form $[0; a + 1, a, 1, a]$ with $a > 1$ even in order ensure we do not have a link. This continued fraction results in $p/q = \frac{(a+1)^2 - 1}{(a+1)^3}$, with $(p+q)|q$.

Subcase 2 Assume, still, that $\partial[0100] \neq \partial[1001]$, but that $\partial[0001]$ does not yield a boundary slope. Then one of $a_0, a_1 = 1$. If $\partial[0010]$ exists, then it must be equal to $\partial[1001]$ as in the previous case, leading to $a_0 = a_3 + a_2$, and $\partial[1000]$ since it would be different from $\partial[0010]$ and $\partial[1001]$, so $a_2 = 1$. This yields a continued fraction of the form $[0; a + 1, 1, 1, a] = \frac{2a+1}{2(a+1)^2}$, which always has an even integer in the denominator, yielding a link rather than a knot.

Subcase 3 In this case, we still assume that $\partial[0100] \neq \partial[1001]$. We further assume that $\partial[0010]$ does not exist, so that $a_0 = 1$, and hence $\partial[0000], \partial[0001]$ also do not exist. Note that if $\partial[0100]$ were to yield a boundary slope, it would be distinct from each other existing boundary slope, so it too must not yield a boundary slope. That is, $a_2 = 1$. This yields a continued fraction of the form $[0; 1, a, 1, b]$ with $b > 1$ and either a even or b odd. This continued fraction yields $p/q = \frac{ab+a+b}{(a+2)(b+1)-1}$ with $(q - p)|(q + 1)$.

Subcase 4 We now assume the $\partial[0100] = \partial[1001]$. That is, $a_3 - a_0 = a_1$. Precisely one additional distinct boundary slope must exist among the substitution patterns $\partial[0000], \partial[0001], \partial[0010], \partial[1000]$. Note that $\partial[0010] < \partial[0100] = \partial[1001]$, hence either $\partial[0010]$ is not a boundary slope, or it yields the fourth boundary slope. In this subcase, we will assume the latter, that $\partial[0010]$ yields a boundary slope. Therefore, $a_0 > 1$. Notice that $\partial[0001]$ must not not exist, since it would be a fifth distinct boundary slope. Hence, $a_1 = 1$, and $\partial[0000]$ also does not exist. The last substitution pattern to consider is $\partial[1000]$; if this pattern does not yield aboundary slope, then $a_2 = 1$, giving us a continued fraction $[0; a - 1, 1, 1, a]$, which represents a link rather than a knot. Therefore, $\partial[1000]$ must exist, and must be equal to $\partial[0010]$, that is $a_0 = a_2$. Since $a_3 - a_0 = a_1$, we get continued fractions of the form $[0; a, 1, a, a + 1]$ with a even. Then $p/q = \frac{(a+1)^2 + 1}{(a+1)^3}$, with $(p-1)|q$.

Subcase 5 We now consider the case where $\partial[0100] = \partial[1001]$ and $\partial[0010]$ does not exist. Hence, $a_3 - a_0 = a_1$ and $a_0 = 1$. Thus, $\partial[0000]$ and $\partial[0001]$ also do not exist. To obtain four boundary slopes, we then require thatn $\partial[1000]$ must exist, so $a_2 > 1$. Note that $\partial[1000]$ is distinct from the other existing boundary slopes, namely $\partial[1001]$. Therefore, we arrive at continued fractions of the form $[0; 1, a, b, a + 1]$ with a even and $b \geq 3$ odd. This results in $p/q = \frac{a^2 b + ab + 2a + 1}{(ab + b + 2)(a + 1)}$, with $(q - p + 1)|q$.

We have now verified every potential subcase for $n = 3$.

Case 3 ($n = 4$): We will show that there is only one way one way for a two bridge knot to have exactly four boundary slopes when $n = 4$. Notice that there $\partial[10101] < \partial[10100] < \partial[10010], \partial[01001] < \partial[01010]$ all exist. Therefore, we require $\partial[10010] = \partial[01001]$, i.e. $-a_0 + a_3 = a_1 - a_4$. Further, note that, if it exists, $\partial[01000]$ would necessarily introduce a fifth boundary slope. Therefore, $a_3 = 1$, also rendering $\partial[00000]$ and $\partial[10000]$ nonexistent. Similarly, $\partial[00101]$ would necessarily introduce a fifth boundary slope, so $a_0 = 1$; this also renders $\partial[00100]$ nonexistent. The only remaining substitution pattern is $\partial[10001]$. If it does not exist, then $a_2 = 1$ and we get a continued fraction of the form $[0; 1, a, 1, 1, a]$, which will have an even denominator and hence does not represent a knot. Therefore, $a_2 > 1$ and $\partial[10001]$ must be equal to $\partial[10100]$, i.e. $-a_0 - a_4 = -a_0 - a_2$, or $a_2 = a_4$. Combining all of these with the equality above, we have $-1 + 1 = a_1 - a_4 = a_1 - a_2$, or $a_1 = a_2 = a_4$. Therefore, we arrive at a continued fraction of the form $[0; 1, a, a, 1, a]$ with $a > 1$ even. This yields $p/q = \frac{a^3 + 2a^2 + a + 1}{(a + 1)^3}$. In this case, $(q - p + 1)|q$.

Case 4 ($n = 5$): In this case, we will show that there is only one simple continued fraction of length $n = 5$ having exactly four boundary slopes. Note that $\partial[101010], \partial[010101], \partial[010100]$ and $\partial[101001]$ will each yield distinct boundary slopes, while $\partial[010010], \partial[100100]$ and $\partial[100101]$ yield some boundary slope. Therefore, we must have that $\partial[100100] = \partial[101001]$, $\partial[100101] = \partial[010100]$, and $\partial[010010] = \partial[101001]$. From these, we get $-a_0 + a_3 = -a_0 - a_2 + a_5$, $-a_0 + a_3 + a_5 = a_1 + a_3$, and $a_1 - a_4 = -a_1 - a_3 + a_5$. Rewritten, $a_5 = a_3 + a_2$, $a_1 = a_5 - a_0$, $a_2 = a_4$. Note that if $\partial[101000]$ or $\partial[000101]$ exist, then a fifth boundary slope is necessarily introduced (they would introduce the second smallest boundary slope, where the smallest is $\partial[101010]$); therefore, $a_0 = a_2 = a_4 = 1$. Lastly, we consider $\partial[010001]$, must be equal to $\partial[010100]$ (implying $a_3 = a_5$, a contradiction since $a_3 = a_4 - a_0$), or it must not exist (implying $a_3 = 1$). Thus, we have $a_0 = a_2 = a_3 = a_4 = 1$, and $a_5 = a_3 + a_2 = 2$, and $a_1 = a_5 - a_0 = 1$. Therefore, the only $n = 5$ simple continued fraction with four distinct boundary slopes is $13/21 = [0; 1, 1, 1, 1, 1, 2]$. Clearly, $21 - 13 - 1 | 21$, that is $q - p - 1|q$.

Case 5 ($n > 5$): In this final case, we show that if $n > 5$, then at least five distinct boundary slopes exist. First, we consider $n = 5$. By Lemma 2, the five boundary slopes $\partial[1010101], \partial[1010100], \partial[1010010], \partial[1001010]$, and $\partial[0101010]$ are distinct. If $n > 5$, by appending $101010 \cdots$ or $010101 \cdots$ to the patterns for $n = 5$, we will have still have at least five distinct boundary slopes.

This completes the proof of Theorem 3. □

CHAPTER 3

Open Questions

In this chapter, we will discuss unanswered questions regarding two–bridge knots and their boundary slopes.

One avenue of interest is the prime factorization of p and q.

QUESTION 1. *What do the prime factorizations of p and q, for a two–bridge knot $K(p/q)$, tell us about the set of boundary slopes of K?*

The Figure–8 knot has boundary slopes $[-4, 0, 4]$, which is a symmetrical set (that is, if c is a boundary slope, then so too is $-c$). For the Figure–8 knot, this is a consequence of the knot being amphichiral. It is evident that any amphichiral two–bridge knot has a symmetric set of boundary slopes, since $K(-p/q)$ is the reflection of $K(p/q)$. However, $K(10/63)$ is not amphichiral (since $10 \not\equiv -10 \pmod{63}$), yet its boundary slopes are symmetric: $[-12, -6, 0, 6, 12]$.

QUESTION 2. *Which two–bridge knots have symmetric sets of boundary slopes?*

Note that $K(1/q), q > 2$ has precisely two boundary slopes, 0 and $2q$, both of which are nonnegative. Another example is $K(5/31)$, with boundary slopes $[0, 12, 22]$. There are many other examples of knots with all nonnegative boundary slopes.

We need not necessarily look at knots with all nonpositive boundary slopes. We need only realize that for a knot $K(p/q)$, its reflection $K(-p/q)$ has as its set of boundary slopes the negation of each boundary slope of $K(p/q)$. For example, $K(-5/31)$ has boundary slopes $[-22, -12, 0]$.

QUESTION 3. *Which two–bridge knots have all boundary slopes nonnegative?*

We have not as yet found an example of a knot with all boundary slopes being positive consecutive integers. though we do have examples of two–bridge knots with boundary slopes following other simple sequences. For example, $K(3/5)$ has boundary slopes $[-4, 0, 4]$ (multiples of 4), and $K(10/63)$ has all boundary slopes a multiple of 6: $[-12, 6, 0, 6, 12]$.

QUESTION 4. *Which two–bridge knots have all boundary slopes being consecutive multiples of $2n$ for some integer n?*

QUESTION 5. *Which two–bridge knots have all boundary slopes being nonnegative consecutive multiples of $2n$ for some integer n?*

QUESTION 6. *We characterized two–bridge knots with 4 or fewer boundary slopes in terms of relationships between numerator and denominator. What about knots with 5 or more boundary slopes?*

In the introduction to Chapter 2, we discussed a relationship between the number of boundary slopes and the existence of a slope of small genus. For example, we showed in Theorem 3 that if a two—bridge knot has 3 boundary slopes then it has a genus 1 boundar slope, but the converse does not hold.

QUESTION 7. *If the knot K has a boundary slope of small genus, does it follow that K has few boundary slopes?*

QUESTION 8. *Do few boundary slopes imply a slope of small genus?*

Ichihara's relationship between crossing number and boundary slopes does not hold for all knots. T. Mattman and Y. Kabaya found several counter–examples among the 3–braid knots. For example, 10_{79} has diameter 22, and 8_{17} has diameter 28. In each of these cases, the diameter is more than twice the crossing number. For 8_{17}, the diameter is actually more than three times the crossing number. However, all of their calculations yield $D(K) \leq 4c(K)$. [**12**][**13**]

QUESTION 9. *For any knot K, is $D(K) \leq 4c(K)$? If not, is there some number n such that for any knot K, $D(K) \leq nc(k)$?*

Ichihara and Shimokawa also provided the result that for a finite boundary slope b of a Montesinos knot K, $|b| \leq 2c(K)$. However, this relationship does not hold for all knots. T. Mattman and Y. Kabaya found several counter–examples among the 3–braid knots, in addition to those discussed above. For example, the smallest boundary slope of the knot 10_{152} is -22, which is in absolute value larger than $2c(K)$.

QUESTION 10. *For a finite boundary slope b of a knot K, is $|b| \leq 4c(K)$? If not, is there some number n such that for any knot K, $D(K) \leq nc(k)$?*

Two–bridge knots of few boundary slopes seem to come from knots that are interesting in other ways. Two–bridge knots with 2 boundary slopes are comprised of torus knots. The set of twist knots is contained within the two–bridge knots with 3 boundary slopes.

QUESTION 11. *What, if any, characterization can be given to the two–bridge knots with 3 boundary slopes which are not twist knots?*

QUESTION 12. *What, if any, characterization can be given to the two–bridge knots with 4 boundary slopes?*

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