COUNTING SPANNING TREES IN GRAPH FAMILIES

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Teresa Mercado

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G - a connected graph

G' - dual graph of *G*

V(G) - vertex set of graph *G*

E(G) - edge set of graph *G*

deg(v) - the degree of vertex *v*

$\kappa(G)$ - the number of spanning trees of graph *G*

a_{ij} - an entry in the i^{th} row and j^{th} column of matrix *A*

A - an adjacency matrix

D - a degree matrix

L - a Laplacian matrix

λ_i - the i^{th} eigenvalue of A

det(L) - the determinant of matrix L

$|V_G|$ - the number of vertices in *G*

$|E_G|$ - the number of edges in *G*

$|F_G|$ - the number of faces in *G*

I - the identity matrix

I_n – the nxn identity matrix

$\mathbf{0}_n$ - the nxn matrix with zeros for all its entries

$\mathbf{1}_n$ - the nxn matrix with ones for all its entries

K_n - Complete Graph Family

K_n^- - Complete Graph minus an Edge Family

C_n- N-Cubed Graph Family

O_n- Octahedral Graph Family

B_n- Bipartite Graph Family

$K_{n,n}$- Complete Bipartite Graph Family

$K_{n,n}^-$- Complete Bipartite Graph minus an Edge Family

K_{2n}/n - Complete Graph Family with $|V| = 2n$ minus n edges

Abstract

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In this thesis we find the number of spanning trees, $\kappa(G)$, for graph families. We begin with well-known families such as the complete graphs K_n, the n-cube graphs C_n and the Platonic Solids. Then, we present two new families, the Octahedral graphs O_n and a family of bipartite graphs, B_n; graphs in these two families are regular. We prove that $\kappa(O_n) = n^{n-2} \cdot n^{(n-2)} \cdot (n-1)^n$ and $\kappa(B_n) = (n-1) \cdot n^{n-2} \cdot (n-2)^{n-1}$ using a Corollary to the Matrix Tree Theorem. For the five graph families, K_n, K_n^-, C_n B_n, and O_n we compare the number of spanning trees by index, proving that $\kappa(K_n^-) < \kappa(K_n) < \kappa(B_n) < \kappa(B_{n+1}) < \kappa(O_n)$ and $\kappa(C_n) \neq \kappa(O_n)$ for $n > 3$. We observe that pairs of graphs in these families have an equal number of spanning trees when they are dual, isomorphic, both disconnected, or both trees and conjecture that these are the only possibilities. Finally, we propose questions for future research including finding the number of spanning trees in families that include graphs with non-regular degree and in pairs of graphs in the five families with the same number of vertices or the same number of edges.

Chapter 1: Introduction

Origins

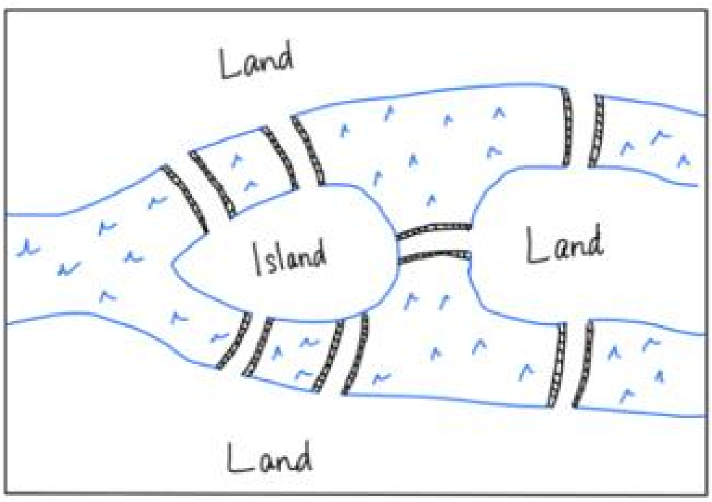
Ever since I completed my first graduate course in Graph Theory with Dr. Mattman, I have been interested in the subject. As I was exploring a research topic for my thesis, I reached out to Dr. Mattman, who had previously offered to advise anyone interested in this branch of mathematics. After meeting with Dr. Mattman I concluded that I wanted to embrace the idea of expanding my knowledge in Graph Theory. I read about the Matrix Tree Theorem in Richard P. Stanley's work, *Enumerative Combinatorics*, which includes a section discussing the number of spanning trees in the family of n-cube graphs. This volume also contains the main result used in this thesis, the Corollary to the Matrix Tree Theorem, which will be presented in Chapter 2. To put my thesis goal in perspective, I delved into finding the spanning trees of another known graph family, the complete graph. I also considered a modification of the complete graph family by removing an edge from each of the complete graphs. Then, I built two different graph families and derived the formula for the number of spanning trees in each. One of the families included the Octahedron, a Platonic Solid, and the second family included the Cube, another Platonic Solid. Since both of the created graph families had a connection with Platonic Solids, I also explored the number of spanning trees of the Platonic Solids. This, in turn suggested investigating if there was any relationship in the number of spanning trees within the graph families, and under what circumstances did these relationships occur.

Background

In 1736, Swiss mathematician Leonhard Euler solved the well-known *Konigsberg Bridge Problem* (Katz and Starbird, 2013). This branch of mathematics, now known as graph theory, began with Euler's solution to that problem. Konigsberg was a city on an island in Prussia, today known as Kaliningrad, see Figure 1, which is surrounded by two rivers and has seven bridges. We follow the presentation of this problem in Katz and Starbird (2013). In the early 1700's, a resident of Konigsberg, Otto, challenged his friend Friedrich to try to leave the island and travel by ground over all seven bridges without crossing over the same bridge twice while returning to the island; hence the name *Konigsberg Bridge Problem*.

Figure 1

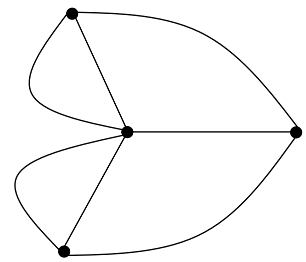
Konigsberg Bridge Problem



Graph Theory is the study of connections between objects. In the case of the *Konigsberg Bridge Problem,* we have the seven bridges as connections, and the four bodies of land as objects. Figure 2 is a graphical representation of the problem in graph theory. The points represent the objects, known as vertices and the curves connecting the points represent the bridges, known as edges. A more complete definition will be stated in Chapter 2. Using graph theory Euler discovered that there was no possible way to travel over all seven bridges without crossing over one of them twice in order to return to the island.

Figure 2

Konigsberg Bridge Problem Graph



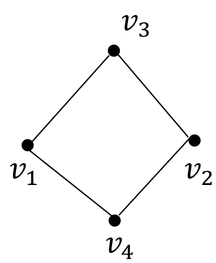
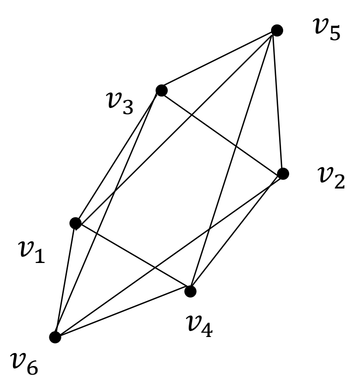
This example shows how graph theory can translate a problem into a simple visual representation. However, in today's society we see more complex applications of graph theory. An example would be social media network analysis. The way individuals are suggested to more friends and followers on social media based on the ones they currently have is analyzed using graph theory. Another example would be how scientists track the spread of diseases. A visual representation of a situation as a graph allows for the creation of solutions in various arrangements such as optimizing, networking, and matching. Therefore, graph theory is the study of relationships between objects.

Statement of Results

We used the Corollary of the Matrix Tree Theorem (Corollary 2.8 in Chapter 2) to find the number of spanning trees in the two families of graphs we created, the Octahedral family and the Bipartite family. For this, we found the adjacency matrix for each of the graphs and its eigenvalues. Using the regular degree, number of vertices, and eigenvalues, (these terms are defined in Chapter 2) we developed a formula that calculates the number of spanning trees for any graph in the Octahedral, O_n, and the Bipartite, B_n, families. For example, Table 1 shows the first few O_n graphs.

Table 1

Octahedral Family, O_n

	Graph	$\boldsymbol{\lvert V \rvert}$	$\boldsymbol{\lvert E \rvert}$	Regular Degree
$\boldsymbol{O_1}$		2	0	0
$\boldsymbol{O_2}$		4	4	2
$\boldsymbol{O_3}$		6	12	3

The following Corollary 3.5 gives the number of spanning trees, denoted $\kappa(G)$, for each graph in this family.

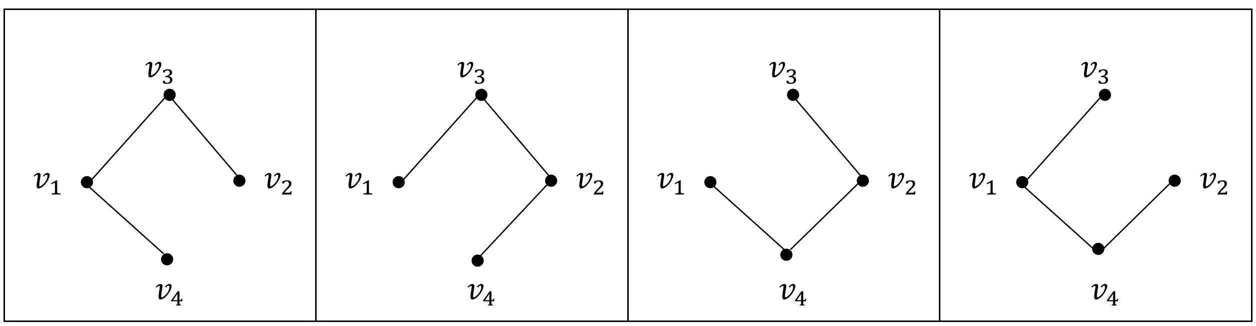
Corollary 3.5 Let $n \geq 1$. The number of spanning trees of O_n is

$$\kappa(O_n) = 2^{2n-2} \cdot n^{n-2} \cdot (n-1)^n.$$

If we consider $n = 1$, $\kappa(O_1) = 2^{2(1)-2} \cdot 1^{1-2} \cdot (1-1)^1 = 2^0 \cdot 1^{-1} \cdot 0^1 = 0$ and this is correct since O_1 has two vertices and no edges. O_1 is a disconnected graph, hence has no spanning trees. Consider O_2: then $n = 2$ we get $\kappa(O_2) = 2^{2(2)-2} \cdot 2^{2-2} \cdot (2-1)^2 = 2^2 \cdot 2^0 \cdot 1^2 = 4$. Figure 3 shows the four spanning trees of O_2.

Figure 3

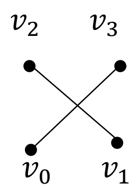
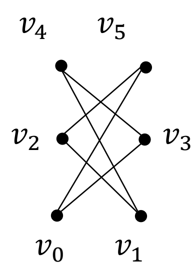
O_2 Spanning Trees



The second graph family created is the Bipartite family. The first few B_n graphs are illustrated in Table 2. It is followed by Corollary 3.10 that quantifies the number of spanning trees for each graph in this family.

Table 2

Bipartite Family, B_n

	Graph	$\|V\|$	$\|E\|$	Regular Degree
B_1		2	0	0
B_2		4	2	1
B_3		6	6	2

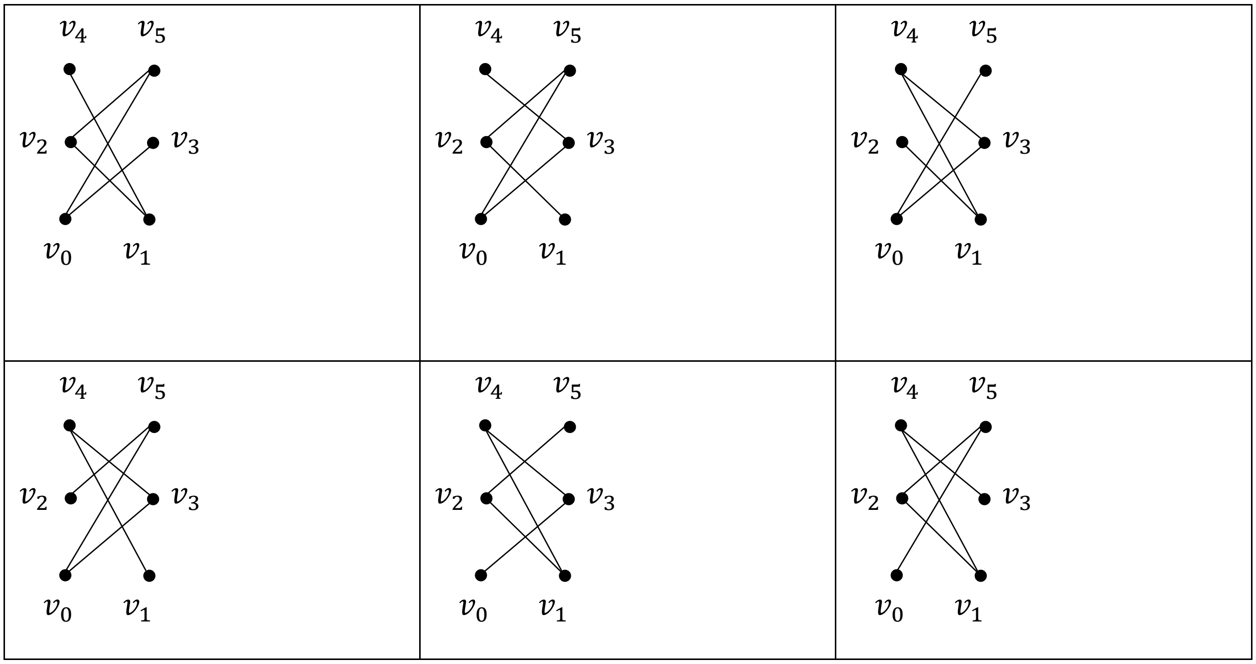
Corollary 3.10 Let $n \geq 1$. The number of spanning trees of $B_n$is

$$\kappa(B_n) = (n-1)(n-2)^{n-1}(n)^{n-2}.$$

Let us calculate κ for the first three graphs of B_n. For both B_1 and B_2 the number of spanning trees is zero since both graphs are disconnected. When $n = 1$, then $\kappa(B_1) = (1-1)(1-2)^{1-1}(1)^{1-2} = 0$ and when $n = 2$, then $\kappa(B_2) = (2-1)(2-2)^{2-1}(2)^{2-2} = 0$. Now B_3 is no longer a disconnected graph; when $n = 3$, $\kappa(B_3) = (3-1)(3-2)^{3-1}(3)^{3-2} = 6$. The Figure 4 shows the six spanning trees for B_3.

Figure 4

B_3 Spanning Trees



In addition to these two new graphs families, we examine three known families, K_n, K_n^- and C_n. We wanted to know when graphs in these families will have the same number of spanning trees. It is known that κ is equal in graphs that are dual or isomorphic, and that disconnected graphs have no spanning trees and that a tree has exactly one spanning tree. For example, all the disconnected graphs, O_1, B_1, B_2, and K_2^-, have no spanning trees. Similarly, all the trees in these graph families have exactly one spanning tree, namely C_1, K_1, K_2, K_1^- and K_3^-.

Conjecture 4.1: If $\kappa(G_1) = \kappa(G_2)$ for graphs in the families K_n, K_n^- C_n, O_n, and B_n, then one of the following must hold: G_1 and G_2 are dual, isomorphic, both disconnected, or both trees.

This conjecture was left for future research, but Theorem 3.15, Theorem 3.20 and Theorem 3.21 were proven in this thesis.

Theorem 3.15: $\kappa(K_n) = \kappa(K_m^-)$ if and only if $n = 1$ or 2 and $m = 1$ or 3. The graphs K_1, K_2, K_1^- and K_3^- are trees and each has one spanning tree.

Theorem 3.20: Let $n > 3$. Then $\kappa(C_n) \neq \kappa(O_n)$.

We prove this theorem by showing that $\kappa(C_n)$ has far more 2's in its prime factorization. Although it was not proven in this thesis, it seems that $\kappa(C_n)$ is much larger than $\kappa(O_n)$.

Theorem 3.21: If $n > 3$, then $\kappa(K_n^-) < \kappa(K_n) < \kappa(B_n) < \kappa(B_{n+1}) < \kappa(O_n)$.

Guide to Thesis

In this chapter we gave an introduction to this thesis, including a statement of the main results of our research. We will now conclude with a summary of the remaining chapters.

In Chapter 2, we introduce the foundational terms and theorems that will guide the reader through the thesis. These are terms and theorems that were known prior to this thesis work. First are the basic definitions of a graph theory course followed by the needed theorems for finding the number of spanning trees in the graph families presented. These are accompanied by illustrations and detailed tables for three graph families, the complete graphs, K_n, the complete graphs minus an edge, K_n^-, and the n-cubes, C_n. This chapter also contains a table with the characteristics of the Platonic Solids including the number of spanning trees. In addition, there are definitions of terms and statements of lemmas and theorems needed in this thesis. Notably, we state the Matrix Tree Theorem and its Corollary, which are the foundation for the development of the formulas for the number of spanning trees in the two constructed graph families. This chapter will conclude with the properties of matrices that help prove both Corollaries 3.5 and 3.10.

In Chapter 3 are the diagrams and explanations of the two new graph families, the Octahedral graph family, O_n, and the Bipartite graph family, B_n. Each is followed by a statement of the number of spanning trees of the graphs in the family: Corollaries 3.5 and 3.10 (stated above). To verify the results, we include calculations of the number of spanning trees for small index and compare the result with isomorphic graphs. Then all five graph families in this thesis are compared to find when they have the same number of spanning trees and what causes this outcome. As a result of this comparison, we prove Theorem 3.21 (stated above). Although the theorem does not include the n-cube graph family, we also prove Theorem 3.20, which shows that the Octahedral graph family, which has largest κ in Theorem 3.21, never has the same number of spanning trees as the n-cube family for a given index $n > 3$.

In Chapter 4 is a final summary of the result of my thesis and suggested questions and conjectures for the further study of spanning trees in graph families. Here we compare graph families that have the same number of vertices and edges at different indexes to see how this can affect the number of spanning trees. The graph families in this thesis all have a unique characteristic; they are all of regular degree. We end by discussing graph families of non-regular degree and how a formula for the number of spanning trees could be derived.

Chapter 2: Literature Review

In this chapter, we define terms and known theorems that will be referenced throughout this thesis. There will also be various figures to expand on the definitions and tables organizing the graphs' characteristics.

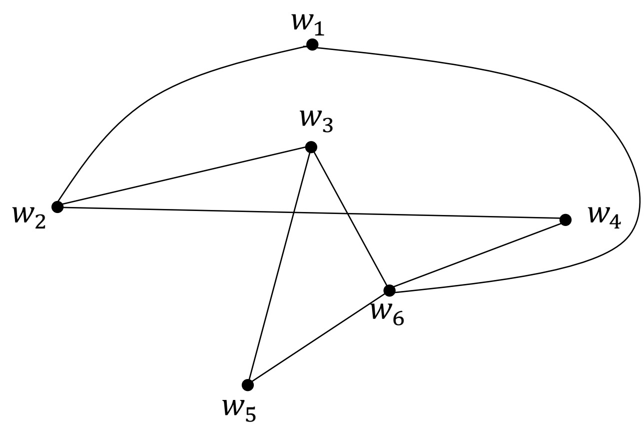
Basic Definitions and Theorems

To determine the number of spanning trees of the new graph families, O_n and B_n, which will be presented in Chapter 3, we must define a few terms used in Graph Theory. The following are basic definitions and theorems about graphs that you can find in any introductory text, for example, *Graph Theory: A Problem Oriented Approach* by Daniel A. Marcus (2008).

A *graph,* G, consists of a set of *vertices,* $V(G)$, and a set of connections, $E(G)$, which are called *edges*, often illustrated as line segments or curves joining pairs of vertices. Formally, an edge is a set of two vertices, $\{v, w\}$, though we will often write vw for short. Vertices that are joined by an edge are called *adjacent vertices*. For example, in Figure 5 $w_1, w_2, \ldots, w_6$ are vertices and w_1w_2 is one edge in the graph. Therefore w_1 and w_2 are adjacent vertices. Graphs will be assumed to be *simple*, meaning there is at most one edge between a given pair of vertices and that there are no loops or edges from a vertex back to itself.

Figure 5

Simple Graph



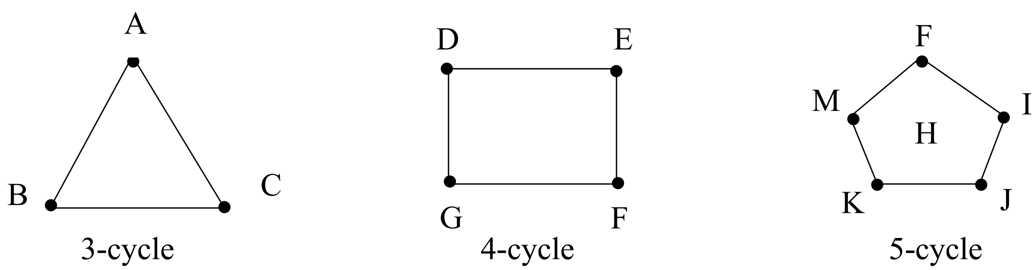
The number of edges with an endpoint at a vertex, w, is the vertex's *degree*, denoted $deg(w)$. For example, in Figure 5 above, $deg(w_6) = 4$ and $deg(w_5) = 2$. When $deg(v) = d$, for all vertices v in a graph G, we say G is *regular, of degree* d or $d - regular$. The graph in Figure 5 above, is not regular since not all the vertices have the same degree. But by using the following Theorem, 2.1, we know that the sum of the degrees of all vertices is 16, since there are eight edges in the graph.

Theorem 2.1: "The Degree Theorem", In any graph, the sum of the degrees of all the vertices is twice the number of edges.

To describe connected graphs, we must first explain paths. An example of a path in Figure 5 above is $w_1, w_1w_6, w_6, w_6w_3, w_3$. A *path* is a sequence of vertices and edges that starts and ends with vertices and alternates between adjacent vertices and connecting edges. A path *connects* the first and last vertex. A graph is a *connected graph* if every vertex is connected to every other vertex of the graph with a path. When a path begins and ends at the same vertex and otherwise repeats no vertices it is called a *cycle*. A graph that is a cycle with *n* vertices and edges is called an *n-cycle* for $n \geq 3$. Figure 6 shows the first three $n - cycle$ graphs.

Figure 6

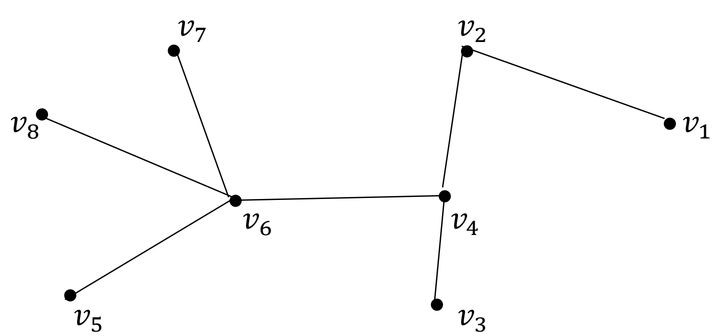
n-cycle Graph



A triangle is a 3-cycle graph, a quadrilateral is a 4-cycle graph, and so forth: hence any *n-gon* can be thought of as an n-cycle graph. A graph is called a *tree* if it is connected and has no cycles. For example, Figure 7 below is a tree. Then the following theorem allows us to determine the number of edges in any given tree.

Figure 7

Tree Graph



Theorem 2.2: “The Tree Theorem”, A tree with *v* vertices has exactly $v - 1$ edges.

By Theorem 2.2 we know that the graph in Figure 7 has seven edges since this graph is a tree with eight vertices, $v_1, v_2, \ldots, v_8$.

In the above graphs, Figures 5 through 7, the vertices were given names or labels. These graphs are called *labeled graphs*. If the vertices are not labeled, then the graph is called an *unlabeled graph*. When two or more labeled graphs have the same pairs of vertices adjacent to each other they are called *equivalent graphs*. Two graphs are *isomorphic* to each other if they can be relabeled in such a way that they become equivalent graphs. In Figure 8 below, the two graphs are not equivalent since AB is an edge on the left graph, but not an edge on the right graph. However, if the graph on the right is relabeled by switching labels for vertices B and C, the two graphs become equivalent graphs. Therefore, the graphs in the Figure 8 are isomorphic.

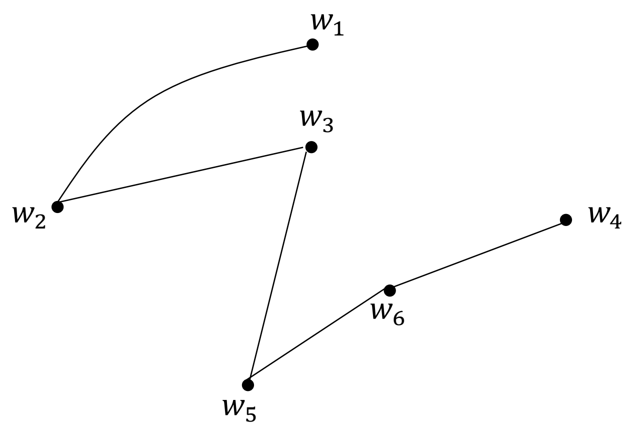
Figure 8

Isomorphic Graphs

If some of the vertices and edges are selected from a graph to create another, the created graph is called a *subgraph*. Hence, a subgraph H of a graph G is a graph where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. When a subgraph contains all the vertices of the original graph in such a way that it creates a tree then it is called a *spanning tree*. So, a spanning tree T of the graph G is a subgraph such that T is a tree and contains all vertices of the graph, $V(T) = V(G)$. For example, the tree in Figure 9 below, is a spanning tree of the graph in Figure 5, since it is a subgraph that is a tree and contains all the vertices.

Figure 9

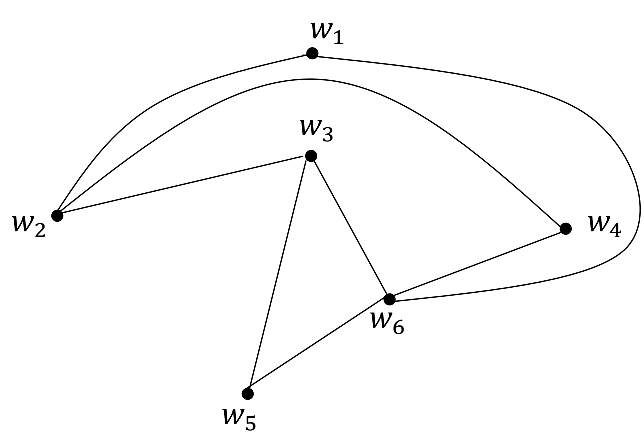
Spanning Tree



A graph G is called *planar* if it can be drawn in the plane ($\mathbb{R}^2$) such that the edges only intersect at the vertices of G. For example, all the above Figures 5 through 9 are planar graphs. We can redraw Figure 5 so that edge w_2w_4 only intersects at the vertices, see Figure 10 below for the diagram.

Figure 10

Planar Graph



Since this graph is drawn in the plane ($\mathbb{R}^2$) it divides $\mathbb{R}^2$ into regions called *faces*. Let *F* denote the number of faces. Each face is bounded by edges which are called *sides* of that face. The region on the outside of a graph is called the *unbounded region*, which is also a face, in Figure 10 above, $|F| = 4$.

Another graph that will be referenced in this thesis is a bipartite graph. A *bipartite graph* is a graph where the vertices can be separated into two disjoint sets, X and Y, where every edge in the graph has one endpoint in each set. When all vertices in one set are adjacent to all vertices in the other set, the graph is complete bipartite. Thus, a *complete bipartite graph* has every vertex in X adjacent to every vertex in Y, and is denoted $K_{m,n}$, where *m* and *n* are the number of vertices in X and Y.

Now that the basic terms are defined, the exploration of spanning trees in graph families K_n, K_n^-, and C_n follows.

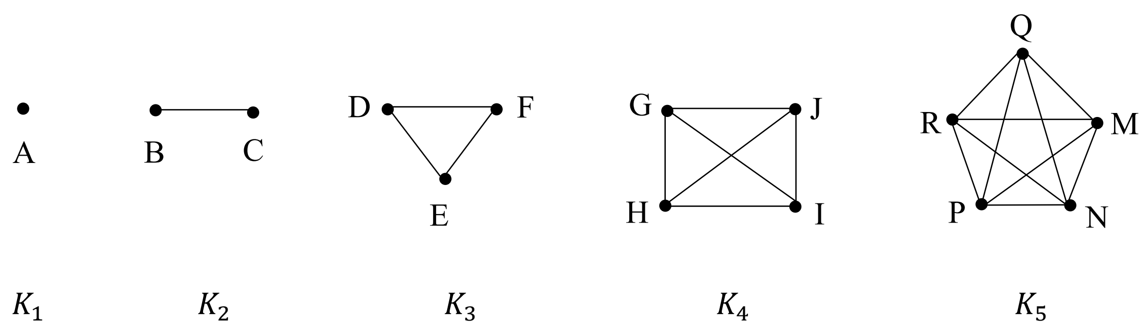
Graph Family $\boldsymbol{K_n}$

A well-known graph family is the complete graphs. A *complete graph* on *n* vertices, denoted K_n, has all n vertices adjacent to each other. Figure 11 contains the first 5 complete graphs. It is known that Cayley's Formula determines the number of spanning trees in a labeled complete graph (Marcus, 2008).

Cayley's Formula If $n \geq 1$, then $\kappa(K_n) = n^{n-2}$.

Figure 11

Complete Graph



Using Cayley's Formula, in Table 3, the number of spanning trees for labeled complete graphs are given.

Table 3

κ for the Complete Graph Family

Complete Graph	Number of Spanning Trees
$\boldsymbol{K_1}$	1
$\boldsymbol{K_2}$	1
$\boldsymbol{K_3}$	3
$\boldsymbol{K_4}$	16
$\boldsymbol{K_5}$	125
⋮	⋮
$\boldsymbol{K_n}$	n^{n-2}

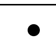
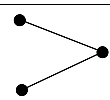
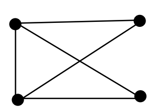
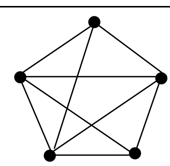
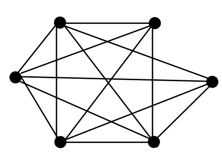
The spanning tree count increases quickly with the number of vertices. The next graph family is created by removing an edge from a labeled complete graph. It is denoted as K_n^-.

Family of Complete Graphs with an Edge Removed, $\boldsymbol{K_n^-}$

Finding the number of spanning trees in K_n^- is a standard exercise in introductory graph theory, for example see Exercise D19 in the book *Graph Theory* (Marcus, 2008), which asks for $\kappa(K_5^-)$. In Table 4 are the illustrations of the first six graphs in the K_n^- family. Note that K_2^- is not connected, hence it has no spanning trees, and K_1^- and K_3^- are trees, therefore they each have one spanning tree.

Table 4

κ and Graphs of K_n^-

	Graph	Number of Spanning Trees
$\boldsymbol{K_1^-}$		1
$\boldsymbol{K_2^-}$		0
$\boldsymbol{K_3^-}$		1
$\boldsymbol{K_4^-}$		8
$\boldsymbol{K_5^-}$		75
$\boldsymbol{K_6^-}$		864
$\vdots$	$\vdots$	$\vdots$
$\boldsymbol{K_n^-}$		$n^{n-3}(n-2)$

According to the *Tree Theorem*, a spanning tree in K_n will have $n - 1$ edges. And by the *Degree Theorem* since K_n is $(n - 1) -$ regular, we can find the number of edges in K_n, as $\frac{n(n-1)}{2}$. Before proving the general result, let us look at some examples. We will follow the probabilistic argument suggested by Marcus (2008). In K_3, the number of edges is three and any spanning tree will contain 2 of those edges. Now let us consider K_3^-: its spanning trees will also have 2 edges. Hence, K_3^- is itself the only spanning tree of K_3^-. Note that this is the same situation for K_1^-: it is its own spanning tree. The graph K_4 has a total of 4^2 or sixteen spanning trees and each will have 3 edges. So, every edge will have a $\frac{3}{6}$ probability of being in a particular spanning tree, since six is the total number of edges of K_4. Therefore, $\kappa(K_4^-) = 8$, one half of the sixteen spanning trees of K_4. The number of spanning trees for K_5 is 125 and each has 4 edges, whereas the complete graph, K_5, will have ten edges. So, any removed edge has a probability $\frac{4}{10}$ of being in a particular spanning tree. Therefore, when deleting an edge, $\frac{2}{5}$ of the spanning trees are removed leaving $\kappa(K_5^-) = 75$, which is $\frac{3}{5}$ of 125. For K_6 the number of spanning trees is 1296, by Cayley's Formula. Each spanning tree will have 5 edges and there are 15 edges in K_6. The edge that is removed has a $\frac{5}{15}$ chance of being in any given spanning tree. Hence $\kappa(K_6^-) = 864$, which is $\frac{10}{15}$ of 1296. Looking at the patterns in this work suggests the following theorem:

Theorem 2.3: Let $n > 1$. The number of spanning trees of the graph K_n^- is

$$\kappa(K_n^-) = n^{n-3}(n - 2).$$

Proof:

The pattern we observed suggests that $\kappa(K_n^-)$ is $\kappa(K_n)$, times one minus the probability of a particular edge being part of a spanning tree. So,

$$\kappa(K_n^-) = (n^{n-2})\left(1 - \frac{n-1}{\frac{n(n-1)}{2}}\right) = n^{n-2}\left(\frac{n-2}{n}\right) = n^{n-3}(n-2).$$

While this probabilistic argument is suggestive, we can also give a more combinatorical proof. In K_n, there are n^{n-2} spanning trees, each having $n-1$ edges. This gives $(n^{n-2})(n-1)$ positions for a particular edge to occupy in a spanning tree of K_n. By symmetry, each edge will occur in the same number of spanning trees. As there are $\frac{n(n-1)}{2}$ edges in K_n, each edge is in $\frac{n^{n-2}(n-1)}{\frac{n(n-1)}{2}} = 2n^{n-3}$ trees. When an edge is deleted, those $2n^{n-3}$ trees are no longer spanning trees and there remain $n^{n-2} - 2n^{n-3} = n^{n-3}(n-2)$ spanning trees, as required.

■

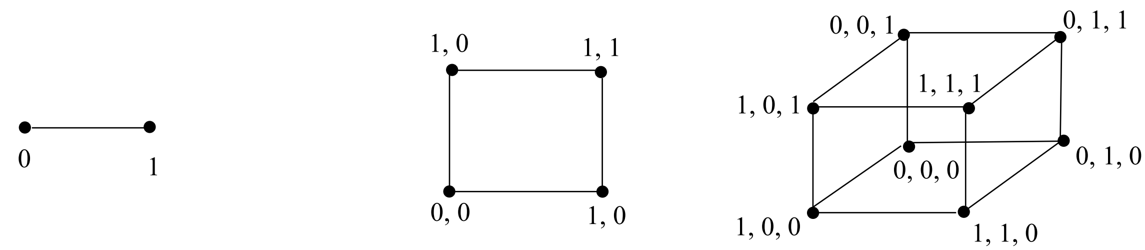
Graph Family C_n

For the next graph family, the n-cubes, we follow Richard P. Stanley's book (1999). The n-cube on 2^n vertices, denoted C_n, is a graph with vertex set the n-tuples of 0's and 1's. Two vertices are adjacent if they differ in exactly one component. Note that C_n is n-regular. Figure 12 below shows the first three n-cube graphs. Notice that C_2 is a cycle, more specifically a 4-cycle. The formula for the number of spanning trees for an n-cube graph family is as follows (Stanley, 1999).

$$\kappa(C_n) = 2^{2^n-n-1}\prod_{i=1}^{n} i^{\binom{n}{i}}$$

Figure 12

Cube Graph



C_1 C_2 C_3

Using this Formula, in Table 5 are the number of spanning trees for the first few n-cube graphs.

Table 5

κ in n-cube Graphs

n-cube	Number of Spanning Trees
$\boldsymbol{C_1}$	1
$\boldsymbol{C_2}$	4
$\boldsymbol{C_3}$	384
$\vdots$	$\vdots$
$\boldsymbol{C_n}$	$2^{2^n-n-1}\prod_{i=1}^{n} i^{\binom{n}{i}}$

We noticed that C_2 is the 4-cycle graph and the number of spanning trees is known for such graphs.

Theorem 2.4: $\kappa(n - cycle) = n$.

Proof:

For an *n*-cycle graph, when an edge is removed a tree is formed. Hence, each spanning tree corresponds to deleting one of the *n* edges.

■

Since C_2 is a 4-cycle, $\kappa(C_2) = 4$. This agrees with $\kappa(C_2)$ as stated in Table 5 above. In the n-cube graphs, the number of spanning trees increases rapidly with the index n, as it did for the complete graph and K_n^- families. In Chapter 3, we introduce the Octahedral family, which includes the Octahedral graph, a Platonic Solid. While exploring this family we realized that C_3 and K_4 are also Platonic Solids. Hence, our next quest was the Platonic Solids.

Platonic Solids

The Platonic Solids are a well know family of regular planar graphs. A *regular planar graph* is a d −regular graph with $d \geq 3$, such that each face is bounded by the same number of edges, again with a minimum of three sides per face, (Katz and Starbird, 2013). A *Platonic Solid,* is a convex, solid object whose faces are flat regular polygons. The edges and vertices of a platonic solid can be identified with a regular planar graph. The following theorem is standard in many introductory graph theory books and also a triumph of Euclid's *Elements* (See Heath and Heiberg, 1908).

Theorem 2.5 "Platonic Solids Theorem." There are exactly five regular planar graphs.

As mentioned above, during this research, it was noticed that K_4 (tetrahedron) and C_3 (cube) are two of the five Platonic Solids. The Table 6 gives some of the characteristics of the five Platonic Solids. The number of spanning trees for K_4 and C_3 is stated in Table 3 and Table 5 respectively. In Chapter 3, the number of spanning trees of the Octahedron, $\kappa(O_3)$, will be determined. SageMath (2020), a computer program, was used to count the number of spanning trees for the remaining two Platonic Solids.

Table 6

Platonic Solids and Characteristics

Platonic Solid	$\|V\|$	$\|E\|$	$\|F\|$	κ
Tetrahedron	4	6	4	16
Cube	8	12	6	384
Octahedron	6	12	8	384
Icosahedron	12	30	20	5,184,000
Dodecahedron	20	30	12	5,184,000

Notice that the Cube and Octahedron have the same number of edges and spanning trees. Also, the number of vertices for the Cube is the same as the number of faces of the Octahedron and vice versa. For the Icosahedron and Dodecahedron, the same scenario happens; they have the same number of edges and spanning trees while the number of vertices in the Icosahedron is the same as the number faces in the Dodecahedron and vice versa. We call these pairs dual graphs. To construct a *dual graph*, G', for G, a connected planar graph, a vertex is placed in each face of G, including the unbounded face; notice $|F_G| = |V_{G'}|$. Now connect each pair of vertices of G' making sure to cross each edge of G only once; these yield $|E_G| = |E_{G'}|$ and $|V_G| = |F_{G'}|$. This now brings us to a well-known theorem in graph theory about the relationship between dual graphs and their number of spanning trees (Biggs, 1971).

Theorem 2.6: Dual graphs have the same number of spanning trees.

This theorem shows why the two pairs of Platonic Solids have the same number of spanning trees.

More Theorems

The following are theorems that will be referenced later in the thesis. First, we define a few matrices (see Stanley, 1999).

For a graph of n vertices, we define

1) the *adjacency matrix* A as an $n \ x \ n$ matrix such that

$$a_{ij} = \begin{cases} 1 \text{ if } w_i \text{ and } w_j \text{ are adjacent} \\ 0 \text{ otherwise} \end{cases}$$

2) the *degree matrix D* as an $n \ x \ n$ diagonal matrix such that

$$d_{ii} = \text{degree of } w_i$$

3) the *Laplacian matrix L* as the degree matrix minus the adjacency matrix

$$L = D - A.$$

A few more matrices that we will use in proofs for this thesis are $\boldsymbol{O}_n$ and $\mathbf{1}_n$. The matrix $\boldsymbol{O}_n$ will be defined as the $n \ x \ n$ matrix whose entries are all zeros and $\mathbf{1}_n$ the $n \ x \ n$ matrix whose entries are all ones.

The main tools in counting the number of spanning trees, are the Matrix Tree Theorem and its Corollary, stated as Theorem 9.8 and Corollary 9.10(b) in Stanley's book (1999).

Theorem 2.7: "The Matrix Tree Theorem", Let *G* be a finite connected graph without loops, with Laplacian matrix ***L***=***L****(G).* Let $\boldsymbol{L_0}$ denote $\boldsymbol{L}$ with the last row and column removed (or with the i^{th} row and column removed for any *i*). Then det($\boldsymbol{L_0}$) = $\kappa(G)$.

The Corollary refers to eigenvalues, so let us define a few more standard terms in linear algebra, for example, see *Elementary Linear Algebra* by Stanley I. Grossman (1994).

Let *M* be an *n x n* matrix with real entries. The complex number λ is called an *eigenvalue* of *M* if there is a nonzero vector $\vec{v}$ in $\mathbb{C}^n$ such that $M\vec{v} = \lambda\vec{v}$. The vector $\vec{v} \neq 0$ is called an *eigenvector* of *M* corresponding to the eigenvalue λ. It is said vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are *linearly independent* provided $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n = 0$ holds only when $a_1 = a_2 = \cdots = a_n = 0$ where $a_i \in \mathbb{R}$.

Corollary 2.8: Suppose that *G* is a connected graph that is regular of degree d with p vertices and that the eigenvalues of the adjacency matrix ***A(G)*** are $\lambda_1, \lambda_2, \dots, \lambda_{p-1}, \lambda_p$ with $\lambda_p = d$. Then $\kappa(G) = \frac{1}{p}(d-\lambda_1)(d-\lambda_2)\cdots\left(d-\lambda_{p-1}\right)$.

To expand on eigenvalues, more definitions and theorems will be stated that can be found in any introductory linear algebra book, for example, Grossman (1994).

Let λ be an eigenvalue for *M* then $M\vec{v} = \lambda\vec{v} = \lambda I\vec{v}$ for some $v \neq 0$.

So $M\vec{v} = \lambda I\vec{v} \Rightarrow M\vec{v} - \lambda I\vec{v} = 0 \Rightarrow (M - \lambda I)v = 0$, with $v \neq 0$.

Hence, $det(M - \lambda I) = 0$ and we call $p(x) = det(M - xI)$ the *characteristic polynomial* of M.

Theorem 2.9: Let M be an $n\ x\ n$ matrix with real entries. Then λ is an eigenvalue of *M* if and only if $p(\lambda) = det(M - \lambda I) = 0$.

If M is an $n\ x\ n$ matrix, the *characteristic polynomial* is a polynomial of degree n. By the Fundamental Theorem of Algebra the *characteristic polynomial*, has exactly n roots (counting multiplicities). Because any eigenvalue of M is a root of the *characteristic equation,* it can be concluded that every $n \times n$ matrix has exactly n eigenvalues. While it will be assumed that the matrix has real entries, in general, the eigenvalues are complex numbers.

Theorem 2.10: Every $n \times n$ matrix with real entries has exactly n eigenvalues, counting multiplicities.

The number of occurrences of λ as a root of $p(x)$ is the *algebraic multiplicity* of λ. The *geometric multiplicity* is the dimension of the eigenspace for λ, the space of eigenvectors for the eigenvalue λ.

Theorem 2.11: Let λ be an eigenvalue of a matrix *M*. Then:

$$\text{the geometric multiplicity of } \lambda \leq \text{the algebraic multiplicity of } \lambda.$$

In this chapter, we have introduced known theorems, definitions, and graph families that we will use to prove different theorems in upcoming chapters.

Chapter 3: Methodology

In this chapter, we develop formulas for the number of spanning trees for the two graph families, the Octahedral family, O_n and the Bipartite family, B_n. This chapter begins by defining and illustrating each family, to then determine its adjacency matrix. This produces the conditions needed to use the Corollary of the Matrix Tree Theorem to find the formula for the spanning trees for each of the graph families.

Inspired by the formula for $\kappa(C_n)$, we set out to find formulas for κ for other graph families. The first graph family we constructed was O_n.

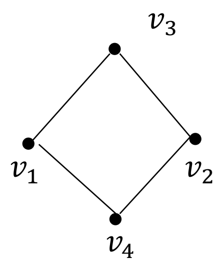
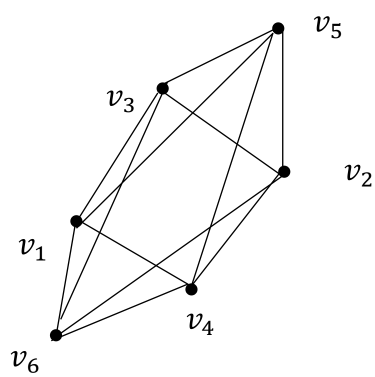
The Octahedral Family, $\mathbf{O_n}$

O_n is inductively defined as a graph on $2n$ vertices. The graph O_n is formed by adding two vertices to O_{n-1} where the two new vertices of O_n are adjacent to all vertices in O_{n-1} but not each other. The graph O_1 begins with two vertices that are not adjacent. Next, O_2 has four vertices: the two additional vertices and the vertices of O_1. The added vertices of O_2 are adjacent to the vertices of O_1 and are not adjacent to each other. So O_2 is a 4-cycle graph, a graph that is a cycle with four vertices. Then O_3 winds up being the Octahedron graph, hence the name Octahedral, O_n, for this graph family. Note that the number of vertices of O_n increases by 2 at each step, and like K_n and C_n these graphs are regular; all vertices have the same degree.

For example, in Table 7 are the first few O_n graphs.

Table 7

O_n and Characteristics

	Graph	$\lvert \boldsymbol{V} \rvert$	$\lvert \boldsymbol{E} \rvert$	Regular Degree
$\boldsymbol{O_1}$		2	0	0
$\boldsymbol{O_2}$		4	4	2
$\boldsymbol{O_3}$		6	12	3
⋮	⋮	⋮	⋮	⋮
$\boldsymbol{O_n}$		*2n*	*2n(n-1)*	*2(n-1)*

Considering both the Matrix Tree Theorem and its Corollary, $\kappa(O_n)$ can be identified using the determinant of L_0 for the Laplacian of O_n or the eigenvalues of the adjacency matrix of O_n. We explored both, the Laplacian matrix and the adjacency matrix of O_n with the SageMath computer program (2020). Since we found patterns in the eigenvalues of the adjacency matrix of O_n, we used the Corollary of the Matrix Tree Theorem.

The following $n\ x\ n$ matrices are needed for Lemma 3.1. Recall that $\mathbf{1}_n$ is the nxn all-ones matrix.

a. I_n is the *identity matrix* such that $(I_n)_{ij} = \begin{cases} 1, & 1 \leq i = j \leq n \\ 0, & else \end{cases}$

b. J_n is a matrix such that $(J_n)_{i,n+i-1} = 1\ for\ 1 \leq i \leq n$ and all other entries are 0.

The identity matrix I_n is defined in every introductory linear algebra textbook. The J_n matrix is similar except that the 1's run along the anti-diagonal from bottom left to top right.

Lemma 3.1: Let $n \geq 1$. The adjacency matrix of O_n is $A_{2n} = \mathbf{1}_{2n} - I_{2n} - J_{2n}$.

Proof by Induction:

For $n = 1$, the adjacency matrix of O_1 is $A_2 = \mathbf{1}_2 - I_2 - J_2$.

$$A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

From the illustration in Table 7 of the graph O_1, the vertices v_1 and v_2 are not adjacent, hence the entries of the adjacency matrix of A_2 are all zeroes. Assume the lemma holds for $n = k$ where $k \geq 1$. That is assume the adjacency matrix of O_k is $A_{2k} = \mathbf{1}_{2k} - I_{2k} - J_{2k}$.

Let $V(O_k) = \{w_1, w_2, \dots, w_{2k}\}$ be the vertices of O_k. Here is the adjacency matrix for O_k:

$$A_{2k} = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 & 0 \\ 1 & 0 & 1 & \dots & 1 & 0 & 1 \\ 1 & 1 & 0 & \dots & 0 & 1 & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & \cdots & 0 & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 0 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0 \end{pmatrix}$$

Notice this matrix is symmetrical in both diagonals and also about the horizontal and vertical midlines.

We want to prove the statement for $n = k + 1$. So, add two more vertices w_0 and w_{2k+1} to O_k that are not adjacent to each other. The matrix A_{2k+2} will be A_{2k} plus two additional rows and columns for vertices w_0 and w_{2k+1} added to the ends. More specifically, the entries for $a_{0,0}$, $a_{2k+1,2k+1}$, $a_{0,2k+1}$ and $a_{2k+1,0}$ will be zero and all other entries in the added rows and columns will be ones. Now $V(O_{k+1}) = \{w_{0,}\ w_1,\ w_2,\ \dots,\ w_{2k},\ w_{2k+1}\}$

and $A_{2k+2} = \begin{pmatrix} 0 & 1 & \cdots & 1 & 0 \\ 1 & & & & 1 \\ \vdots & & \boxed{\quad A_{2k} \quad} & & \vdots \\ 1 & & & & 1 \\ 0 & 1 & \cdots & 1 & 0 \end{pmatrix}$

Since we know $A_{2k} = \mathbf{1}_{2k} - I_{2k} - J_{2k}$ then $A_{2k+2} = \mathbf{1}_{2k+2} - I_{2k+2} - J_{2k+2}$ is proven as required. Therefore, by mathematical induction $A_{2n} = \mathbf{1}_{2n} - I_{2n} - J_{2n}$ is the adjacency matrix of O_n.

■

In light of Corollary 2.8, we need the eigenvalues of the adjacency matrix of O_n to calculate $\kappa(O_n)$. To find them we used the SageMath computer program (2020) and found patterns in the eigenvalues. Theorem 3.4 below yields the eigenvalues of the adjacency matrix of O_n which are needed to apply Corollary 2.8 to find $\kappa(O_n)$. But first, we prove Lemma 3.2, from which Lemma 3.3 follows trivially.

Let $\vec{c}_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ be the *constant one's vector* with n entries.

Lemma 3.2: Let $n \geq 1$. If $\vec{v} = (v_1,\ v_2, \dots, v_n)^T$ then

a. $\mathbf{1}_n\vec{v} = (\sum_{i=1}^{n} v_i)\vec{c}_n$

b. $I_n\vec{v} = \vec{v}$

c. $J_n\vec{v} = (v_n,\ v_{n-1}, \dots, v_1)^T$

Proof:

a. $$\mathbf{1}\vec{v} = \begin{pmatrix} 1(v_1) +\ 1(v_2) + \cdots + 1(v_n) \\ 1(v_1) +\ 1(v_2) + \cdots + 1(v_n) \\ \vdots \\ 1(v_1) +\ 1(v_2) + \cdots + 1(v_n) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} v_i \\ \sum_{i=1}^{n} v_i \\ \vdots \\ \sum_{i=1}^{n} v_i \end{pmatrix}$$

$$= \sum_{i=1}^{n} v_i \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \left(\sum_{i=1}^{n} v_i\right)\vec{c}_n$$

b. For $I_n\vec{v} = \vec{v}$ see any Linear Algebra Book, for example, Grossman (1994).

c. $$J_n\vec{v} = \begin{pmatrix} 0(v_1) +\ 0(v_2) + \cdots + 1(v_n) \\ 0(v_1) +\ 0(v_2) + \cdots + 0(v_n) \\ \vdots \\ 1(v_1) +\ 0(v_2) + \cdots + 0(v_n) \end{pmatrix} = \begin{pmatrix} v_n \\ v_{n-1} \\ \vdots \\ v_1 \end{pmatrix} = (v_n,\ v_{n-1}, \dots, v_1)^T$$

■

Define the *standard basis* of $\mathbb{R}^n$: $\vec{e}_i$ for $(1 \leq i \leq n)$ where $(\vec{e}_i)_j = \begin{cases} 1 & for\ i = j \\ 0 & else \end{cases}$. Then we have the following, which is immediate from Lemma 3.2c.

Lemma 3.3: Let $n \geq 1$. Then $J_n\vec{e}_i = \vec{e}_{n-i+1}$ for $1 \leq i \leq n$.

Theorem 3.4: Let $n \geq 1$. The eigenvalues of A_{2n}, the adjacency matrix of O_n, are $2(n-1)$, -2, and 0 with algebraic and geometric multiplicities of one, $n-1$, and n, respectively.

Proof:

We will show $\vec{c}$ is an eigenvector. We know $A = \mathbf{1} - I - J$.

So $A\vec{c} = \mathbf{1}\vec{c} - I\vec{c} - J\vec{c}$

$= 2n\vec{c} - \vec{c} - \vec{c}$ by Lemma 3.2

$= 2n\vec{c} - 2\vec{c}$

$= 2(n-1)\vec{c}$

Hence $\lambda = 2(n-1)$ is an eigenvalue for the eigenvector $\vec{c}$.

Let $\vec{v}_i = \vec{e}_i + \vec{e}_{2n-i+1} - \vec{e}_n - \vec{e}_{n+1}$ for $1 \leq i \leq n-1$. We will show that each $\vec{v}_i$ is an eigenvector. By Lemma 3.2 and 3.3 we know $J\vec{v}_i = \vec{v}_i$ and since $\sum_{k=1}^{2n}(\vec{v}_i)_k = 0$ and $\mathbf{1}\vec{v}_i = 0$,

$A\vec{v}_i = \mathbf{1}\vec{v}_i - I\vec{v}_i - J\vec{v}_i$

$= 0\vec{c} - \vec{v}_i - \vec{v}_i$

$= -2\vec{v}_i$

Hence $\lambda = -2$ is an eigenvalue for each eigenvector $\vec{v}_i$.

Let $\vec{u}_i = \vec{e}_i - \vec{e}_{2n-i+1}$ for $1 \leq i \leq n$. We will show that each $\vec{u}_i$ is an eigenvector. By Lemma 3.2 and 3.3 we know $J\vec{u}_i = -\vec{u}_i$ and since $\sum_{k=1}^{2n}(\vec{u}_i)_k = 0$ and $\mathbf{1}\vec{u}_i = 0$,

$A\vec{u}_i = \mathbf{1}\vec{u}_i - I\vec{u}_i - J\vec{u}_i$

$= 0\vec{c} - \vec{u}_i - (-\vec{u}_i)$

$= -\vec{u}_i - (-1)\vec{u}_i$

$= -\vec{u}_i + \vec{u}_i$

$= 0\vec{u}_i$

Hence $\lambda = 0$ is an eigenvalue for each eigenvector $\vec{u}_i$.

Therefore, we have found the following eigenvalues for A_{2n}: $2(n-1)$ with eigenvector $\vec{c}$, -2 with eigenvectors $\vec{v}_1$, $\vec{v}_2, \dots, \vec{v}_{n-1}$, and 0 with eigenvectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$.

Now we must show that the geometric multiplicity of -2 is at least $n-1$ by proving $\vec{v}_1$, $\vec{v}_2, \dots, \vec{v}_{n-1}$ are linearly independent. That is, we will show

$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_{n-1}\vec{v}_{n-1} = 0$ implies all $a_i = 0$. Recall $\vec{v}_i = \vec{e}_i + \vec{e}_{2n-i+1} - \vec{e}_n - \vec{e}_{n+1}$ for $1 \le i \le n-1$. Assume $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_{n-1}\vec{v}_{n-1} = 0$.

$$0 = \sum_{i=1}^{n-1} a_i\, \vec{v}_i \quad \text{where } n > 1$$

$$= \sum_{i=1}^{n-1} a_i(\vec{e}_i + \vec{e}_{2n-i+1} - \vec{e}_n - \vec{e}_{n+1})$$

$$= \sum_{i=1}^{n-1} (a_i\vec{e}_i + a_i\vec{e}_{2n-i+1} - a_i\vec{e}_n - a_i\vec{e}_{n+1})$$

$$= \sum_{i=1}^{n-1} a_i\vec{e}_i + \sum_{i=1}^{n-1} a_i\, \vec{e}_{2n-i+1} - \sum_{i=1}^{n-1} a_i\vec{e}_n - \sum_{i=1}^{n-1} a_i\, \vec{e}_{n+1}$$

Thus

$$0 = \left(a_1, a_2, a_3, \dots, a_{n-1}, -\sum_{i=1}^{n-1} a_i\, , -\sum_{i=1}^{n-1} a_i\, , a_{n-1}, \dots, a_3, a_2, a_1,\right)$$

To get the zero vector, we must have $a_i = 0$ for all $1 \leq i \leq n-1$. Therefore $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}$ are linearly independent vectors.

Now we must show the geometric multiplicity of 0 is at least n by proving $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ are linearly independent. That is, we will show $a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_n\vec{u}_n = 0$ implies all $a_i = 0$. Recall $\vec{u}_i = \vec{e}_i - \vec{e}_{2n-i+1}$ for $1 \leq i \leq n$. Assume $a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_n\vec{u}_n = 0$.

$$0 = \sum_{i=1}^{n} a_i\,\vec{u}_i$$

$$= \sum_{i=1}^{n} a_i(\vec{e}_i - \vec{e}_{2n-i+1})$$

$$= \sum_{i=1}^{n} (a_i\vec{e}_i - a_i\vec{e}_{2n-i+1})$$

$$= \sum_{i=1}^{n} a_i\vec{e}_i - \sum_{i=1}^{n} a_i\,\vec{e}_{2n-i+1}$$

Thus, $0 = (a_1, a_2, a_3, \dots, a_n, -a_n, \dots, -a_3, -a_2, -a_1,)$. To get the zero vector, we must have $a_i = 0$ for all $1 \leq i \leq n$. Therefore $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ are linearly independent vectors.

Now we want to show that we have found all the eigenvalues for A_{2n}. The sum of the geometric multiplicities for A_{2n} is at least $1 + n - 1 + n = 2n$ since the multiplicities of the eigenvalues $2(n-1)$, -2, and 0 are at least 1, $n-1$ and n respectively. By Theorem 2.10 the adjacency matrix A_{2n} has exactly $2n$ eigenvalues, therefore the sum of the algebraic multiplicities is $2n$. And by Theorem 2.11 the geometric multiplicity $\lambda \leq$ the algebraic multiplicity of λ.

It follows that:

$2n \leq$ the sum of the geometric multiplicities of $2(n-1), -2,$ and 0

$\leq$ the sum of the algebraic multiplicities of $2(n-1), -2,$ and $0 \leq 2n$.

For this to be true all the inequalities must be equal signs. Therefore $2n$ equals the sum of the geometric multiplicities of the three eigenvalues that are equal to the sum of the algebraic multiplicities of the three eigenvalues which are equal to $2n$ and, by Theorem 2.10, we have found all the eigenvalues of A_{2n}. $\blacksquare$

Now that we know all the eigenvalues of the adjacency matrix for O_n we can determine the number of spanning trees.

Corollary 3.5: Let $n \geq 1$. The number of spanning trees of $O_n$is:

$$\kappa(O_n) = 2^{2n-2} \cdot n^{n-2} \cdot (n-1)^n.$$

Proof:

Since O_1 is not connected it has no spanning trees. Our formula for $\kappa(O_n)$ gives the correct value zero. For the remainder of the proof, we assume $n > 1$.

The graph O_n is connected and $2(n-1) -$ regular with $|V(O_n)| = 2n$. Its eigenvalues are $2(n-1)$ with a multiplicity of one, -2 with a multiplicity of $n-1$, and 0 with a multiplicity of n. Corollary 2.8 states that $\kappa(G) = \frac{1}{p}(d-\lambda_1)(d-\lambda_2)\cdots(d-\lambda_{p-1})$, and we know that $p = 2n$, $d = 2(n-1)$, $\lambda_{1,1} = -2$, $\lambda_{1,2} = -2, \cdots, \lambda_{1,n-1} = -2$, and $\lambda_{2,1} = 0, \lambda_{2,2} = 0, \cdots, \lambda_{2,n} = 0$. Thus, by this corollary we have:

$$\kappa(O_n) = \frac{1}{2n}(2(n-1) - (-2))^{n-1}(2(n-1))^n$$

$$= \frac{1}{2n}(2n-2+2)^{n-1}(2^n(n-1)^n)$$

$$= \frac{1}{2n}(2n)^{n-1}(2^n(n-1)^n)$$

$$= (2n)^{-1}(2n)^{n-1}(2^n(n-1)^n)$$

$$= (2n)^{n-2}(2^n(n-1)^n)$$

$$= (2)^{n-2}(n)^{n-2}(2^n(n-1)^n)$$

$$= 2^{2n-2} \cdot n^{n-2} \cdot (n-1)^n$$

Therefore $\kappa(O_n) = 2^{2n-2} \cdot n^{n-2} \cdot (n-1)^n$.

■

Let us calculate the spanning trees for the first few O_n graphs using Corollary 3.5 and verify this value by looking at isomorphic graphs of each O_n or the graphs of O_n themselves in the Table 7. For O_1, $\kappa(O_1) = 2^{2(1)-2} \cdot (1)^{1-2} \cdot (1-1)^1 = 2^0 \cdot 1^{-1} \cdot 0^1 = 0$ which is correct. Since O_1 is not connected, it has no spanning trees. This is shown in Table 7. For O_2, $\kappa(O_2) = 2^{2(2)-2} \cdot (2)^{2-2} \cdot (2-1)^2 = 2^2 \cdot 2^0 \cdot 1^2 = 4$ which is correct because O_2 is isomorphic to C_2, which has four spanning trees. This is shown in the Table 5. These two graphs are also isomorphic to the 4-cycle graph and, by Theorem 2.4, $\kappa(n-cycle) = n$. Therefore $\kappa(4-cycle) = 4$, the same as both $\kappa(O_2)$ and $\kappa(C_2)$. For O_3, $\kappa(O_3) = 2^{2(3)-2} \cdot (3)^{3-2} \cdot (3-1)^3 = 2^4 \cdot 3^1 \cdot 2^3 = 384$ which is correct. Recall that we found κ for each of the

Platonic solids in Chapter 2, including the Octahedron graph, O_3. In particular, we observed that $\kappa(O_3)$ should agree with that of its dual graph, the cube.

After finding the formula for $\kappa(O_n)$, we noticed that O_n is the same graph as K_{2n} / n edges, that is K_{2n} with n edges removed. For K_{2n} / n edges to be isomorphic to O_n we must remove n edges in such a way that we have $deg(v) = 2(n-1)$ for all vertices of K_{2n} / n edges. This means that no pair of removed edges share a vertex. We will elaborate more on this topic in Chapter 4 under questions for future research.

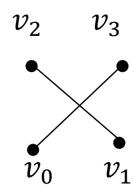
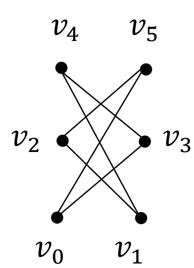
We set out to explore more graph families and were influenced by the K_n, K_n^- and K_{2n} / n edge families. The second graph family we constructed was $B_n = K_{n,n}/\, n$ edges, a bipartite graph.

Bipartite Graph Family B_n

$B_n = K_{n,n}/\, n$ edges was created by adding two non-adjacent vertices to $B_{n-1} = K_{n-1,n-1}/$ $(n-1)$ edges. Let $V_1 = \{w_1, w_2, \dots, w_{n-1}\}$ and $V_2 = \{w_{n+1}, w_{n+2}, \dots, w_{2n-2}\}$ be the partition of V(B_{n-1}). We add the vertex w_{2n-1} adjacent to all vertices in V_1 and the vertex w_0 adjacent to those in V_2. Then B_n has $2n$ vertices and $n^2 - n$ edges. So B_1 has 2 vertices and no edges, and B_2 has 4 vertices and 2 edges. The added vertices of B_2 are adjacent to certain vertices of B_1 but not to each other. Notice that the number of vertices increases by 2 for every subsequent B_n . And again, like graphs K_n, C_n and O_n, B_n is regular; all vertices have the same degree. Table 8 shows the first few B_n graphs.

Table 8

B_n and Characteristics

	Graph	$\lvert V \rvert$	$\lvert E \rvert$	Regular Degree
$\boldsymbol{B_1}$		2	0	0
$\boldsymbol{B_2}$		4	2	1
$\boldsymbol{B_3}$		6	6	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\boldsymbol{B_n}$		$2n$	$n^2 - n$	$n - 1$

Our goal is to find $\kappa(B_n)$ using Corollary 2.8. The first step is Lemma 3.7 below, which gives the adjacency matrix of B_n. Let us define H as a $2n\ x\ 2n$ block matrix with the top right and bottom left $n\ x\ n$ blocks being the all ones matrix $\mathbf{1}_n$ and the top left and bottom right $n\ x\ n$ blocks being the all zero matrix $\mathbf{0}_n$ as follows:

$$H_{2n} = \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix}.$$

Lemma 3.7: Let $n \geq 1$. The adjacency matrix for B_n is $A_{2n} = H_{2n} - J_{2n}$.

Proof by Induction:

For $n = 1$, then $A_2 = H_2 - J_2$.

$$A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

From Table 8, the graph B_1, has the vertices v_0 and v_1 that are not adjacent, hence the matrix shows no adjacent vertices. Assume the lemma holds for $n = k$ where $k \geq 1$. That is, assume the adjacency matrix for B_k is $A_{2k} = H_{2k} - J_{2k}$.

Let $V(B_k) = \{w_1, w_2, \ldots, w_{2k}\}$ be the vertices of B_k. Here is A_{2k}, the adjacency matrix for B_k:

$$A_{2k} = \begin{pmatrix} & & & \cdots & 1 & 1 & 0 \\ & \mathbf{0}_k & & \cdots & 1 & 0 & 1 \\ & & & \cdots & 0 & 1 & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & \cdots & & & \\ 1 & 0 & 1 & \cdots & & \mathbf{0}_k & \\ 0 & 1 & 1 & \cdots & & & \end{pmatrix}$$

Notice this matrix has identical blocks. The top left and bottom right are both the all zeros matrix $\mathbf{0}_k$. And the blocks at the top right and bottom left also match. Let $V_1 = \{w_i \text{ with } 1 \leq i \leq k\}$ and $V_2 = \{w_j \text{ with } k+1 \leq j \leq 2k\}$. No pair of vertices in V_1 are adjacent to one another and also no pair from V_2 are adjacent; hence the blocks of zeros on the top left and bottom right. The entries along the main and anti-diagonals are zeros since there are no loops and k edges are deleted, compare to $K_{k,k}$. The ones entries in A_{2k} represent the adjacent vertices between V_1 and V_2.

We want to prove the statement for $n = k + 1$. We add two more vertices w_0 and w_{2k+1} to B_k which are not adjacent to each other. The matrix A_{2k+2} will be A_{2k} plus two additional rows and columns, vertex w_{2k+1} added to V_2 and vertex w_0 to V_1 at the ends. More specifically the entries for $a_{0,0}$, $a_{2k+1,2k+1}$, $a_{0,2k+1}$ and $a_{2k+1,0}$ will be zero. Since w_0 is adjacent to vertices in V_2 and w_{2k+1} is adjacent to vertices in V_1 this accounts for the ones in the added rows and columns. And $V(B_{k+1}) = \{w_{0,}\ w_1, w_2, \ldots, w_{2k}, w_{2k+1}\}$.

Thus $$A_{2k+2} = \begin{pmatrix} 0 & 0 & \cdots & 1 & 0 \\ 0 & \mathbf{0's} & & & 1 \\ \vdots & & A_{2k} & & \vdots \\ 1 & & & \mathbf{0's} & 0 \\ 0 & 1 & \cdots & 0 & 0 \end{pmatrix}$$

We know $A_{2k} = H_{2k} - J_{2k}$, so then $A_{2k+2} = H_{2k+2} - J_{2k+2}$ as required. Therefore, by mathematical induction, $A_{2n} = H_{2n} - J_{2n}$ is the adjacency matrix for B_n.

■

To find $\kappa(B_n)$, we need the eigenvalues of the adjacency matrix for B_n so we can use Corollary 2.8. We embarked on finding these using the SageMath computer program (2020), and observed patterns in the eigenvalues of the adjacency matrix for B_n. The result was Theorem 3.9 that gives the eigenvalues of the adjacency matrix for B_n. We will first prove Lemma 3.8 about matrices and vectors which is needed in the proof for Theorem 3.9.

Lemma 3.8: Let $n \geq 1$. If $\vec{v} = (v_1, v_2, \dots, v_{2n})^T$ then $(H_{2n}\vec{v})_j = \begin{cases} \sum_{i=1}^{n} v_{n+i} & for \quad 1 \leq j \leq n \\ \sum_{i=1}^{n} v_i & for \;\; n+1 \leq j \leq 2n \end{cases}$.

Proof:

$$H_{2n}\vec{v} = \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{2n-1} \\ v_{2n} \end{pmatrix}$$

$$= \begin{pmatrix} v_{n+1} + \cdots + v_{2n} \\ v_{n+1} + \cdots + v_{2n} \\ \vdots \\ v_1 + \cdots + v_n \\ v_1 + \cdots + v_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} v_{n+i} \\ \sum_{i=1}^{n} v_{n+i} \\ \vdots \\ \sum_{i=1}^{n} v_i \\ \sum_{i=1}^{n} v_i \end{pmatrix} = \left(\sum_{i=1}^{n} v_{n+i}, \sum_{i=1}^{n} v_{n+i}, \cdots \sum_{i=1}^{n} v_i, \sum_{i=1}^{n} v_i \right)^T$$

■

Let us define $\vec{d}_{2n}$ as a vector of $2n$ entries where half of the entries are one and the other half are negative one. That is, the ith entry of $\vec{d}_{2n} = \begin{cases} 1 & for \quad 1 \leq i \leq n \\ -1 & for \; n+1 \leq \; i \leq 2n \end{cases}$.

Theorem 3.9: Let $n \geq 1$. The eigenvalues of A_{2n} for B_n are $(n-1)$, $-(n-1)$, -1, and 1 with algebraic and geometric multiplicities of 1, 1, $(n-1)$ and $(n-1)$ respectively.

Proof:

We will show $\vec{c}$ is an eigenvector. Recall $\vec{c}$ is the all one's vector. We know $A = H - J$.

So $A\vec{c} = H\vec{c} - J\vec{c}$. By Lemma 3.2 we know $J\vec{c} = \vec{c}$ and by Lemma 3.8 $H\vec{c} = n\vec{c}$ since $\sum_{i=1}^{n} c_{n+i} = n$ and $\sum_{i=1}^{n} c_i = n$.

Then $A\vec{c} = n\vec{c} - \vec{c}$

$= (n - 1)\vec{c}$.

Hence $\lambda = (n-1)$ is an eigenvalue for the eigenvector $\vec{c}$. Now we will show $\vec{d}$ is an eigenvector. We know $A = H - J$. By Lemma 3.2 we know $J_n\vec{d} = -\vec{d}$ and by Lemma 3.8 $H\vec{d} = -n\vec{d}$ since $\sum_{i=1}^{n} d_{n+i} = -n$ and $\sum_{i=1}^{n} d_i = n$.

So $A\vec{d} = H\vec{d} - J\vec{d}$

$= -n\vec{d} - (-\vec{d})$

$= -n\vec{d} + \vec{d}$

$= (-n+1)\vec{d}$

$= -(n-1)\vec{d}$

Hence $\lambda = -(n-1)$ is an eigenvalue with the eigenvector $\vec{d}$.

Let $\vec{w}_i = \vec{e}_i - \vec{e}_n + \vec{e}_{n+1} - \vec{e}_{2n-i+1}$ for $1 \leq i \leq n-1$. We will show that each $\vec{w}_i$ is an eigenvector. We know $A = H - J$. By Lemma 3.2 we know $J\vec{w}_i = -w_i$ and by Lemma 3.8 $H\vec{w}_i = \vec{0}$ since $\sum_{j=1}^{n}(w_i)_{n+j} = 0$ and $\sum_{j=1}^{n}(w_i)_j = 0$.

So $A\vec{w}_i = H\vec{w}_i - J\vec{w}_i$

$= 0 - (-w_i)$

$= w_i$

Hence $\lambda = 1$ is an eigenvalue for each eigenvector $\vec{w}_i$.

Let $\vec{v}_i = \vec{e}_i - \vec{e}_n - \vec{e}_{n+1} + \vec{e}_{2n-i+1}$ for $1 \leq i \leq n-1$. We will show that each $\vec{v}_i$ is an eigenvector. By Lemma 3.2 we know $J\vec{v}_i = \vec{v}_i$ and by Lemma 3.8 $H\vec{v}_i = \vec{0}$ since $\sum_{j=1}^{n}(\vec{v}_i)_j = 0$ and $\sum_{j=1}^{n}(\vec{v}_i)_{j+n} = 0$.

$A\vec{v}_i = H\vec{v}_i - J\vec{v}_i$

$= \vec{0} - \vec{v}_i$

$= -\vec{v}_i$

Hence $\lambda = -1$ is an eigenvalue for each eigenvector $\vec{v}_i$.

Now that we have the eigenvalues of A_{2n} for B_n, we will show they have the following geometric multiplicities: $(n-1)$ with a multiplicity of 1, $-(n-1)$ with a multiplicity of 1, -1 with a multiplicity of $(n-1)$ and 1 with a multiplicity of $(n-1)$.

For this, we must show the vectors $\vec{w}_i$ and $\vec{v}_i$ are linear independent. That is, we show that the geometric multiplicity of -1 is at least $n-1$ and 1 is $(n-1)$ by proving the eigenvector$s$ are linearly independent. We already showed the $\vec{v}_i's$ are independent in our proof of Theorem 3.4. The proof for the $\vec{w}_i's$ is similar and we will omit it.

Now we want to show that we have found all the eigenvalues for A_{2n}. The sum of the geometric multiplicities for A_{2n} is at least $1 + 1 + n - 1 + n - 1 = 2n$ since the multiplicities of the eigenvalues $(n-1)$, $-(n-1), -1$ and 1 are at least 1, 1, $n-1$ and $n-1$ respectively. By Theorem 2.10 the adjacency matrix A_{2n} has exactly $2n$ eigenvalues, therefore the sum of the algebraic multiplicities is $2n$. And by Theorem 2.11 the geometric multiplicity $\lambda \leq$ the algebraic multiplicity of λ.

It follows that:

$2n \leq$ the sum of the geometric multiplicities $(n-1)$, $-(n-1), -1$ and $1 \leq$ the sum of the algebraic multiplicities of $(n-1)$, $-(n-1), -1$ and $1 \leq 2n$.

For this to be true all the inequalities must be equal signs. Therefore $2n$ equals the sum of the geometric multiplicities of the four eigenvalues that are equal to the sum of the algebraic multiplicities of the four eigenvalues which is equal to $2n$ and, by Theorem 2.10, we have found all the eigenvalues of A_{2n}.

■

Now that we know all the eigenvalues of the adjacency matrix for B_n we can determine the formula for the number of spanning trees.

Corollary 3.10: Let $n \geq 1$. The number of spanning trees of B_n is:

$$\kappa(B_n) = (n-1)(n-2)^{n-1}(n)^{n-2}.$$

Proof:

Since B_1 and B_2 are not connected graphs they have no spanning trees. Our formula for $\kappa(B_n)$ works for these cases as it gives the value of zero. For the rest of the proof, we assume $n > 2$.

The graph B_n is connected and $(n-1)-$ regular with $|V(B_n)| = 2n$. Its eigenvalues are $(n-1)$ with a multiplicity of one, $-(n-1)$ with a multiplicity of one, 1 with a multiplicity of $(n-1)$, and -1 with a multiplicity of $(n-1)$. Corollary 2.8 states that

$\kappa(G) = \frac{1}{p}(d-\lambda_1)(d-\lambda_2)\cdots(d-\lambda_{p-1})$, and we know that $p = 2n$, $d = (n-1)$, $\lambda_{1,1} = -(n-1)$, $\lambda_{2,1} = 1, \lambda_{2,1} = 1, \cdots, \lambda_{2,n-1} = 1$, and $\lambda_{3,1} = -1, \lambda_{3,2} = -1, \cdots, \lambda_{3,n-1} = -1$, thus by Corollary 2.8 we have:

$$\kappa(B_n) = \frac{1}{2n}((n-1)-(-(n-1)))\big((n-1)-1\big)^{n-1}\big((n-1)-(-1)\big)^{n-1}$$

$$= \frac{1}{2n}(2n-2)(n-2)^{n-1}(n)^{n-1}$$

$$= \frac{1}{2n}(2(n-1))(n-2)^{n-1}(n)^{n-1}$$

$$= \frac{1}{n}(n-1)(n-2)^{n-1}(n)^{n-1}$$

$$= (n-1)(n-2)^{n-1}(n)^{n-2}$$

Therefore $\kappa(B_n) = (n-1)(n-2)^{n-1}(n)^{n-2}$.

■

As mentioned before B_1 and B_2 are not connected, see Table 8, and they have no spanning trees. Let us show this using $\kappa(B_n)$. For B_1 we have $\kappa(B_1) = (1-1)(1-2)^{1-1}(1)^{1-2} = 0$ as predicted. And for B_2 we have $\kappa(B_2) = (2-1)(2-2)^{1-1}2^{2-2} = 0$ as suggested earlier. Let us look at B_3. Using Corollary 3.10, $\kappa(B_3) = (3-1)(3-2)^{3-1}3^{3-2} = 2(1^2)(3) = 6.$ Now let us use Corollary 2.8 of the Matrix Tree Theorem to verify the value of

$\kappa(B_3)$. The eigenvalues for B_3 are 2, -2, 1, and -1 whose regular degree is two and total number of vertices is six. Thus $\kappa(B_3) = \frac{1}{6}(2-(-2))(2-1)(2-(-1)) = \frac{1}{6}(4)(1)(3) = \frac{1}{6}(12) = 6$. These two formulas yield the same number of spanning trees. Another observation to make is that B_3 is isomorphic to the 6-cycle graph and, by Theorem 2.4, $\kappa(n - Cycle) = n$. Therefore, $\kappa(6 - Cycle) = 6$, the same as $\kappa(B_3)$. Finally, B_4 is isomorphic to C_3 and B_4 is dual to O_3. Both have 384 spanning trees. Let us calculate $\kappa(B_4)$ with Corollary 3.10,

$\kappa(B_4) = (4-1)(4-2)^{4-1}4^{4-2} = 3(2^3)(4^2) = 3(8)(16) = 384$, as anticipated.

We have presented five graph families in this thesis K_n, K_n^-, C_n, O_n, and B_n. Some of these families have isomorphic graphs between them and others have dual graphs. We will investigate when graphs in these families have the same number of spanning trees and describe relationships between the graph families.

Comparing Graph Families

Throughout the creation and exploration of these graph families we encountered similarities and disparities amongst them. We decided to explore when, if ever, do graphs in the families have the same number of spanning trees. It is interesting to note that each of the graph families have different number of spanning trees as n increases. But there are some n values for which $\kappa(G_n)$ are the same between graph families either because they are isomorphic graphs or because they are dual graphs. Next, we will compare the formulas for the number of spanning trees in graph families and note where these have the same value. In Table 9 below are the formulas for $\kappa(G_n)$ for all graph families. Notice that all formulas contain Cayley's Formula except for $\kappa(C_n)$. We will elaborate more on this observation, in Chapter 4 under questions for future research.

Table 9

Graph Families Formulas for κ

Name	$\boldsymbol{\kappa}$ Formula
Complete Graphs	$\kappa(K_n) = n^{n-2}$, for $n \geq 1$
Complete Graphs minus an edge	$\kappa(K_n^-) = n^{n-2} \cdot \left(1 - \frac{2}{n}\right)$, for $n > 1$
n-cube	$\kappa(C_n) = 2^{2^n-n-1} \prod_{i=1}^{n} i^{\binom{n}{i}}$ for $n \geq 1$
Octahedral	$\kappa(O_n) = n^{n-2} \cdot 2^{2n-2} \cdot (n-1)^n$ for $n \geq 1$
Bipartite	$\kappa(B_n) = n^{n-2} \cdot (n-1) \cdot (n-2)^{n-1}$ for $n \geq 1$

Lemma 3.11: If $n > 1$, then $\kappa(K_n) < \kappa(O_n)$. If $n > 2$, then $\kappa(K_n) < \kappa(B_n)$.

Proof: The numbers $\kappa(O_n)$ and $\kappa(B_n)$ are $\kappa(K_n)$ multiplied by a natural number.

■

Our next goal is to show that the graph families generally do not have common values of $\kappa(G_n)$ and to point out the exceptions where they do have the same $\kappa(G_n)$ for some small values of n.

Let us begin with K_n and K_n^-. Table 10 below shows the numbers of spanning trees for the first few $\kappa(K_n)$ and $\kappa(K_n^-)$.

Table 10

κ for K_n and K_n^-

$\boldsymbol{n}$	$\boldsymbol{\kappa(K_n)}$	$\boldsymbol{\kappa(K_n^-)}$
1	1	1
2	1	0
3	3	1
4	16	8
$\vdots$	$\vdots$	$\vdots$
$\boldsymbol{n}$	n^{n-2}, for $n > 0$	$n^{n-2} \cdot \left(1 - \frac{2}{n}\right)$, for $n > 1$

From Table 10, $\kappa(K_1) = \kappa(K_1^-) = \kappa(K_2) = \kappa(K_3^-)$. These graphs are all trees. We will show that $\kappa(K_n) \neq \kappa(K_m^-)$ except for these four trees.

Lemma 3.12: If $m > 1$, then $\kappa(K_m^-) < \kappa(K_m)$.

Proof:

Recall that $\kappa(K_n) = n^{n-2}$. For $m > 1$,

$$\kappa(K_m^-) = m^{m-2} \cdot \left(1 - \frac{2}{m}\right) = \kappa(K_m) \cdot \left(1 - \frac{2}{m}\right) < \kappa(K_m).$$

■

Lemma 3.13: The sequence $\kappa(K_n)$ is increasing. In fact, $\kappa(K_{n+1}) \geq 3\kappa(K_n)$ for $n > 1$.

Proof by Induction:

For $n = 2$, $\kappa(K_3) \geq 3\kappa(K_2)$ since $\kappa(K_3) = 3$ and $\kappa(K_2) = 1$, as required. Assume the lemma is true for $n = k > 1$: $\kappa(K_{k+1}) \geq 3\kappa(K_k)$.

Prove $n = k + 1$ is true.

$$\kappa\left(K_{(k+1)+1}\right) = \kappa(K_{k+2})$$

$$= (k+2)^k$$

$$= (k+2)^k \cdot \frac{\kappa(K_{k+1})}{\kappa(K_{k+1})}$$

$$= \frac{(k+2)^k}{(k+1)^{k-1}} \cdot \kappa(K_{k+1})$$

$$= (k+2) \cdot \left(\frac{k+2}{k+1}\right)^{k-1} \cdot \kappa(K_{k+1})$$

$$> (k+2) \cdot \kappa(K_{k+1})$$

$$> 3\kappa(K_{k+1}).$$

Therefore $\kappa(K_{k+2}) > 3\kappa(K_{k+1})$. By mathematical induction $\kappa(K_{n+1}) \geq 3\kappa(K_n)$ for $n >$ 1. ■

Recall that $\kappa(K_1) = \kappa(K_1^-) = \kappa(K_2) = \kappa(K_3^-)$. We will argue that this is the only time they agree. First, we show the following lemma.

Lemma 3.14: Let $n > 1$ and $m > 2$. If $\kappa(K_n) = \kappa(K_m^-)$, then $n = m - 1$.

Proof:

By Lemma 3.12 $\kappa(K_m) > \kappa(K_m^-)$ for $m > 1$. Since both sequences are increasing, if $\kappa(K_n) = \kappa(K_m^-)$, we must have $n < m$.

By Lemma 3.13, $\kappa(K_{n+1}) \geq 3\kappa(K_n)$ for $n > 1$. Since $m > 2$, $\left(1 - \frac{2}{m}\right) \geq \frac{1}{3}$ and

$$\kappa(K_m^-) = m^{m-2} \cdot \left(1 - \frac{2}{m}\right) = \kappa(K_m) \cdot \left(1 - \frac{2}{m}\right) \geq \frac{1}{3}\kappa(K_m).$$

Then, if $\kappa(K_n) = \kappa(K_m^-)$, we have

$$\kappa(K_n) = \kappa(K_m^-) \geq \frac{1}{3}\kappa(K_m) \geq \kappa(K_{m-1}).$$

This shows $n \geq m - 1$. Together this gives $m > n \geq m - 1$, which forces $n = m - 1$.

■

Theorem 3.15: $\kappa(K_n) = \kappa(K_m^-)$ if and only if $n = 1$ or 2 and $m = 1$ or 3. The graphs K_1, K_2, K_1^-, and K_3^- are trees and each has one spanning tree.

Proof:

If $n = 1$, K_1 is a tree and $\kappa(K_1) = 1$. For K_m^-, the trees are K_1^- and K_3^- with $\kappa(K_1^-) = 1$ and $\kappa(K_3^-) = 1$. So, we can assume $n > 1$. For $m = 2$, K_2^- is a disconnected graph and $\kappa(K_2^-) = 0$. There are no disconnected graphs among the K_n. If $m = 1$, K_1^- is a tree and K_1, and K_2, are the only trees in the K_n family, so, we can assume $m > 2$.

By Lemma 3.14, we can assume $n = m - 1$. Now, equating $\kappa(K_m^-)$ and $\kappa(K_{m-1})$:

$$m^{m-2} \cdot \left(1 - \frac{2}{m}\right) = (m-1)^{m-1-2}$$

gives $\left(1 - \frac{1}{m}\right)^{m-3} = m - 2$. This is true when $m = 3$. If $m > 3$, $\left(1 - \frac{1}{m}\right)^{m-3} < 1 < m - 2$ which shows $\kappa(K_{m-1}) \neq \kappa(K_m^-)$ except when $m = 3$.

∎

Now let us look at B_n and O_n. Recall that $\kappa(B_n) = (n-1) \cdot n^{n-2} \cdot (n-2)^{n-1}$ and $\kappa(O_n) = 2^{2n-2} \cdot n^{n-2} \cdot (n-1)^n$. Table 11 below gives the numbers of spanning trees for B_n and O_n for small n.

Table 11

κ for B_n and O_n

$\boldsymbol{n}$	$\boldsymbol{\kappa(B_n)}$	$\boldsymbol{\kappa(O_n)}$
1	0	0
2	0	4
3	6	384
4	384	82,944
5	40,500	32,768,000
6	6,635,520	20,736,000,000
7	1,575,656,250	19,271,206,310,000
⋮	⋮	⋮
$\boldsymbol{n}$	$(n-1) \cdot n^{n-2} \cdot (n-2)^{n-1}$	$2^{2n-2} \cdot n^{n-2} \cdot (n-1)^n$

For the finite set $1 \leq m \leq n \leq 7$, observe that $\kappa(B_n) = \kappa(O_m)$ for $\kappa(B_1) = \kappa(B_2) = \kappa(O_1)$ and $\kappa(B_4) = \kappa(O_3)$. None of B_1, B_2 and O_1 are connected graphs, hence all have zero spanning trees. And both B_4 and O_3 are platonic solids and dual graphs of each other; B_4 is isomorphic to the cube and O_3 is the Octahedral graph, see Chapter 2.

These examples suggest we compare B_{n+1} with O_n. Now,

$$\kappa(B_{n+1}) = (n+1-1)\cdot(n+1)^{n+1-2}\cdot(n+1-2)^{n+1-1} = (n)\cdot(n+1)^{n-1}\cdot(n-1)^n$$

and $\kappa(O_n) = 2^{2n-2}\cdot n^{n-2}\cdot(n-1)^n$.

Lemma 3.16: For $n > 3$, $\kappa(B_{n+1}) < \kappa(O_n)$.

Proof:

First observe that $n^2 < \left(\frac{4n}{n+1}\right)^{n-1}$ for $n > 3$. Indeed, the left side is a quadratic, and the right side grows exponentially (For $n > 3$, $\left(\frac{4n}{n+1}\right)^{n-1} > 3^{n-1}$). This show that the right side dominates for large n and it's easy to verify that the inequality holds for $n > 3$. And with some simple manipulation (shown below) we can deduce from this that

$$2^{2n-2}\cdot n^{n-2}\cdot(n-1)^n > n\cdot(n+1)^{n-1}\cdot(n-1)^n.$$

This shows $\kappa(B_{n+1}) < \kappa(O_n)$ because $\kappa(B_{n+1}) = (n)\cdot(n+1)^{n-1}\cdot(n-1)^n$ and $\kappa(O_n) = 2^{2n-2}\cdot n^{n-2}\cdot(n-1)^n$.

$$\text{For } n > 3, n^2 < \left(\frac{4n}{n+1}\right)^{n-1}$$

$$=> \frac{1}{n^2} > \left(\frac{n+1}{4n}\right)^{n-1}$$

$$=> \frac{(4n)^{n-1}}{n^2} > (n+1)^{n-1}$$

$$=> \frac{4^{n-1}n^{n-1}}{n} > n(n+1)^{n-1}$$

$$=> \ 2^{2n-2} \cdot n^{n-2} > n \cdot (n+1)^{n-1}$$

$$=> \ 2^{2n-2} \cdot n^{n-2} \cdot (n-1)^n > n \cdot (n+1)^{n-1} \cdot (n-1)^n$$

Therefore, $\kappa(B_{n+1}) < \kappa(O_n)$ for $n \ > \ 3$.

■

Lemma 3.17: $\kappa(B_n)$ is increasing. In fact, $\kappa(B_{n+1}) \geq 64\kappa(B_n)$ for $n > 2$.

Proof by Induction:

For $n = 3$, $\kappa(B_4) \geq 64\kappa(B_3)$ since $\kappa(B_4) = 384$ and $\kappa(B_3) = 6$, as required.

Similarly, it's easy to verify the lemma for $n \ = \ 4, 5$, and 6 by direct calculation. Assume the lemma is true for $n = k > 5$: $\kappa(B_{k+1}) \geq 64\kappa(B_k)$.

Prove $n = k + 1$ is true.

$$\kappa\left(B_{(k+1)+1}\right) = \kappa(B_{k+2})$$

$$= (k+2-1) \cdot (k+2)^{k+2-2} \cdot (k+2-2)^{k+2-1}$$

$$= (k+1) \cdot (k+2)^k \cdot (k)^{k+1} \cdot \frac{\kappa(B_{k+1})}{\kappa(B_{k+1})}$$

$$= \frac{(k+1) \cdot (k+2)^k \cdot (k)^{k+1}}{(k) \cdot (k+1)^{k-1} \cdot (k-1)^k} \cdot \kappa(B_{k+1})$$

$$= \left(\frac{1}{k+1}\right)^{k-2} \cdot \left(\frac{k(k+2)}{k-1}\right)^{k} \cdot \kappa(B_{k+1})$$

$$= \left(\frac{1}{k+1}\right)^{k-2} \cdot (k+2)^{k} \cdot \left(\frac{k+2}{k+2}\right)^{-2} \cdot \left(\frac{k}{k-1}\right)^{k} \cdot \kappa(B_{k+1})$$

$$= \left(\frac{1}{k+1}\right)^{k-2} \cdot (k+2)^{k} \cdot \frac{(k+2)^{-2}}{(k+2)^{-2}} \cdot \left(\frac{k}{k-1}\right)^{k} \cdot \kappa(B_{k+1})$$

$$= \left(\frac{1}{k+1}\right)^{k-2} \cdot (k+2)^{k-2} \cdot (k+2)^{2} \cdot \left(\frac{k}{k-1}\right)^{k} \cdot \kappa(B_{k+1})$$

$$= \left(\frac{k+2}{k+1}\right)^{k-2} \cdot (k+2)^{2} \cdot \left(\frac{k}{k-1}\right)^{k} \cdot \kappa(B_{k+1})$$

$$> (k+2)^{2} \cdot \kappa(B_{k+1})$$

$$\geq 64\kappa(B_{k+1}).$$

Therefore $\kappa(B_{k+2}) > 64\kappa(B_{k+1})$ for $k > 5$. And by mathematical induction $\kappa(B_{n+1}) \geq 64\kappa(B_n)$ for $n > 2$.

■

Lemma 3.18: For $n > 1$, $\kappa(O_n) > \kappa(B_n)$.

Proof:

From Lemma 3.17, $\kappa(B_{n+1}) \geq 64\kappa(B_n)$ for $n > 2$. And from Lemma 3.16, $\kappa(B_{n+1}) < \kappa(O_n)$ for $n > 3$. So $\kappa(O_n) > \kappa(B_n)$ for $n > 3$. We can see from Table 11 that this also holds true for $n = 2$ and 3.

■

Our last pair of graph families to compare is C_n and O_n. Recall that $\kappa(C_n) = 2^{2^n-n-1}\prod_{i=1}^{n} i^{\binom{n}{i}}$ and $\kappa(O_n) = 2^{2n-2} \cdot n^{n-2} \cdot (n-1)^n$. Table 12 below list the first few spanning tree counts for these two graph families.

Table 12

κ for O_n and C_n

$\boldsymbol{n}$	$\boldsymbol{\kappa(O_n)}$	$\boldsymbol{\kappa(C_n)}$
1	0	1
2	4	4
3	384	384
4	82,944	42,467,330
$\vdots$	$\vdots$	$\vdots$
$\boldsymbol{n}$	$2^{2n-2} \cdot n^{n-2} \cdot (n-1)^n$	$2^{2^n-n-1}\prod_{i=1}^{n} i^{\binom{n}{i}}$

For the finite set $1 < n \leq 4$ we can see that $\kappa(O_2) = \kappa(C_2)$ and $\kappa(O_3) = \kappa(C_3)$. Both O_2 and C_2 are the same cycle graph with 4 vertices and 4 edges. By Theorem 2.4, $\kappa(n - cycle) = n$, we have $\kappa(4 - cycle) = 4$. While O_3 is the octahedral graph and C_3 is the cube graph of the platonic solids, these two are dual graphs, and Theorem 2.6 states dual graphs have the same number of spanning trees. We will show that $\kappa(O_n) \neq \kappa(C_n)$ for $n > 3$.

Lemma 3.19: Let $n > 4$. Then $2^n - n - 1 > 2n - 2 + n(log_2 n)$.

Proof:

An exponential function grows faster than a linear function so, the lemma holds for sufficiently large n. It's easy to check that it will hold for $n > 4$.

■

Theorem 3.20: Let $n > 3$. Then $\kappa(C_n) \neq \kappa(O_n)$.

Proof

Recall that $\kappa(C_n) = 2^{2^n-n-1}\prod_{i=1}^{n} i^{\binom{n}{i}}$ and $\kappa(O_n) = 2^{2n-2} \cdot n^{n-2} \cdot (n-1)^n$. Let us consider the power of 2 in the prime factorization of both $\kappa(C_n)$ and $\kappa(O_n)$ to show that $\kappa(C_n)$ will have more 2's. We are grateful to Dr. McGown for suggesting this approach. The power of the 2 for $\kappa(C_n)$ is at least $2^n - n - 1$.

We will argue that $2n - 2 + n\, log_2 n$ is an upper bound for the power of 2 in $\kappa(O_n)$. If n is even, $n = 2^k(2l + 1)$ where $k > 0$ and $l \geq 0$, then the power of the 2 in $\kappa(O_n)$ is $2n - 2 + k(n - 2)$. From $n = 2^k(2l + 1)$ we know that $k = log_2 \frac{n}{(2l+1)}$ and $k = log_2 n - log_2(2l + 1)$ so $k < log_2 n$. Thus $2n - 2 + n(log_2 n) > 2n - 2 + kn > 2n - 2 + k(n - 2)$.

If n is odd, $n - 1 = 2^k(2l + 1)$ where $k > 0$ and $l \geq 0$, then the power of the 2 for $\kappa(O_n)$ is $2n - 2 + kn$. Again $k < log_2(n - 1) < log_2 n$, so $2n - 2 + n(log_2 n) > 2n - 2 + kn$

By Lemma 3.19, $2^n - n - 1 > 2n - 2 + n(log_2 n)$ for all $n > 4$, so $\kappa(C_n) \neq \kappa(O_n)$. When $n = 4$, Table 12 gives the required result.

Therefore $\kappa(C_n) \neq \kappa(O_n)$ for $n > 3$.

■

Theorem 3.21: If $n > 3$, then $\kappa(K_n^-) < \kappa(K_n) < \kappa(B_n) < \kappa(B_{n+1}) < \kappa(O_n)$

Proof:

Summarizing the above we have $\kappa(K_n^-) < \kappa(K_n)$, (for $n > 1$) by Lemma 3.12, $\kappa(K_n) < \kappa(B_n)$, (for $n > 2$) by Lemma 3.11 and $\kappa(B_n) < \kappa(B_{n+1}) < \kappa(O_n)$, (for $n > 3$) by Lemmas 3.16 through 3.18. $\blacksquare$

Among our five graph families, there are very few examples of pairs of graphs with the same κ. For example, for graphs with the same index n, we have found only the five following examples: $\kappa(K_1) = \kappa(C_1) = \kappa(K_1^-)$, $\kappa(C_2) = \kappa(O_2)$, $\kappa(B_2) = \kappa(K_2^-)$, $\kappa(B_1) = \kappa(O_1)$, and $\kappa(C_3) = \kappa(O_3)$. Table 13 summarizes κ for all five graph families for $n \leq 4$.

Table 13

κ for K_n, K_n^-, C_n, B_n, and O_n

n	$\boldsymbol{\kappa(K_n)}$	$\boldsymbol{\kappa(K_n^-)}$	$\boldsymbol{\kappa(C_n)}$	$\boldsymbol{\kappa(B_n)}$	$\boldsymbol{\kappa(O_n)}$
1	1	1	1	0	0
2	1	0	4	0	4
3	3	1	384	6	384
4	16	8	$4.246733x10^7$	384	82,944

Table 13 shows these five are the only examples with $n \leq 3$, and Theorems 3.20 and 3.21 almost give a proof for the $n > 3$ case. What's missing to complete the argument is a proof that $\kappa(C_n) > \kappa(O_n)$ for $n > 3$. The list of examples of graphs with possibly different index and the same κ is a bit longer. We found the following $\kappa(K_1) = \kappa(K_2) = \kappa(K_1^-) = \kappa(K_3^-) = \kappa(C_1)$, $\kappa(K_2^-) = \kappa(O_1) = \kappa(B_1) = \kappa(B_2)$, and $\kappa(B_4) = \kappa(C_3) = \kappa(O_3)$. We return to this idea in the next chapter where we conjecture that these examples are a complete list of instances of graphs in these families that have the same spanning tree count.

Chapter 4: Results and Future Research

Summary of results

To find the number of spanning trees in graph families O_n and B_n, we used the Corollary to the Matrix Tree Theorem. The following are the results of the research for this thesis.

Synopsis: Counting Spanning Trees in Graph Families:

i. $\kappa(O_n) = 2^{2(n-1)} \cdot n^{n-2} \cdot (n-1)^n$ (Corollary 3.5),

ii. $\kappa(B_n) = (n-1) \cdot n^{n-2} \cdot (n-2)^{n-1}$ (Corollary 3.10),

iii. $\kappa(K_n) = \kappa(K_m^-)$ if and only if $n = 1, 2$, and $m = 1, 3$ (Theorem 3.15),

iv. $\kappa(C_n) \neq \kappa(O_n)$ for $n > 3$ (Theorem 3.20),

v. $\kappa(K_n^-) < \kappa(K_n) < \kappa(B_n) < \kappa(B_{n+1}) < \kappa(O_n)$ for $n > 3$ (Theorem 3.21),

vi. For graphs with the same index, $\kappa(K_1) = \kappa(C_1) = \kappa(K_1^-)$, $\kappa(C_2) = \kappa(O_2)$, $\kappa(B_2) = \kappa(K_2^-)$, $\kappa(B_1) = \kappa(O_1)$, and $\kappa(C_3) = \kappa(O_3)$ and

vii. For graphs with a possibly different index, $\kappa(K_1) = \kappa(K_2) = \kappa(K_1^-) = \kappa(K_3^-) = \kappa(C_1)$, $\kappa(K_2^-) = \kappa(O_1) = \kappa(B_1) = \kappa(B_2)$, and $\kappa(B_4) = \kappa(C_3) = \kappa(O_3)$.

Suggestions for Future Research

Considering these results, we propose some questions for future research. The first question is to verify the suggestion that all the graph in the five families with the same κ have been identified in Synopsis vi and vii. We then ask how κ is related to the number of vertices or edges.

Graphs with the same κ.

We know that graphs have equal κ when they are dual, isomorphic, disconnected, or trees. This leads us to the following conjecture.

Conjecture 4.1: If $\kappa(G_1) = \kappa(G_2)$ for graphs in the families K_n, K_n^-, C_n, O_n, and B_n then one of the following must hold: $G_1$and G_2 are dual, isomorphic, both disconnected, or both trees.

Although we could not prove the conjecture, Synopsis iii to vii above are partial results that support it. Below we list all the instances in the families of graphs with the same κ.

A. Isomorphic graphs:

There are two different types of isomorphic graphs, connected and disconnected.

I. Connected

$\kappa(K_1) = \kappa(K_1^-) = 1$, both are a single vertex.

$\kappa(C_1) = \kappa(K_2) = 1$, the same graph made of an edge and two vertices

$\kappa(C_2) = \kappa(O_2) = 4$, C_2 and O_2 are both 4-cycle graphs.

$\kappa(B_4) = \kappa(C_3) = 384$, the 3-cube graph.

II. Disconnected

$\kappa(O_1) = \kappa(B_1) = \kappa(K_2^-)$, O_1, B_1 and K_2^- are 2 nonadjacent vertices.

B. Dual Graphs

$\kappa(B_4) = \kappa(C_3) = \kappa(O_3) = 364$, C_3 and B_4 are the cube and O_3 is the octahedron graph, which are both Platonic solids and dual to one another.

C. Disconnected graphs

$\kappa(K_2^-) = \kappa(O_1) = \kappa(B_1) = \kappa(B_2) = 0$ because all are disconnected graphs.

D. Trees

$\kappa(K_1) = \kappa(K_2) = \kappa(K_1^-) = \kappa(K_3^-) = \kappa(C_1) = 1$; these are all trees.

Conjecture 4.2: The examples in Synopsis iii, vi and vii are a complete list of graphs in the five families K_n, K_n^-, C_n, O_n, and B_n with the same number of spanning trees.

Note that Conjecture 4.2 implies Conjecture 4.1. It is easy to check Conjecture 4.2 for $n \leq 3$, see Table 13.

Conjecture 4.3: The examples of Synopsis vi are a complete list of graphs in the five families K_n, K_n^-, C_n, O_n, and B_n where two graphs with the same index have the same number of spanning trees.

It is easy to verify Conjecture 4.3 for $n \leq 3$, see Table 13. Using Theorem 3.21, the following conjecture would imply Conjecture 4.3.

Conjecture 4.4: $\kappa(C_n) > \kappa(O_n)$ for $n > 3$.

The evidence to support this is Theorem 3.20 where it is proven that $\kappa(C_n) \neq \kappa(O_n)$. In the proof, we show that $\kappa(C_n)$ has more 2's in its prime factorization than $\kappa(O_n)$. Table 12 also supports Conjecture 4.4 by showing the large difference in value for $\kappa(O_4)$ and $\kappa(C_4)$. Combining Conjecture 4.4 and Theorem 3.21 would show that no pair of graphs with the same index $n > 3$ have the same number of spanning trees. Therefore Conjecture 4.4 implies Conjecture 4.3.

Conjecture 4.4 combined with Theorem 3.21 shows that for a given index n, $\kappa(C_n)$ is larger than $\kappa(G_n)$ for any G_n in $\{K_n, K_n^-, O_n, B_n\}$. We think this is likely due to the number of vertices, see Table 14 below.

Table 14

Vertices in Graph Families

Graph Family	$\boldsymbol{|V|}$
$\boldsymbol{K_n}$	n
$\boldsymbol{K_n^-}$	n
$\boldsymbol{C_n}$	2^n
$\boldsymbol{O_n}$	$2n$
$\boldsymbol{B_n}$	$2n$

We can see that the number of vertices of C_n grows exponentially. This led us to delve into the relationship between κ and the number of vertices in the graph families. And later to also look at κ and the number of edges in the families.

Comparing κ in Graph Families When $|V|$ is the Same

All other graph families have a linear number of vertices, n or $2n$, while C_n has an exponential number of vertices, 2^n. This contributes to the rapid growth of $\kappa(C_n)$ and explains why it is, apparently, the largest of them all for fixed $n > 3$. But let us compare the graph families when the number of vertices is the same. Let $n = 2^k$ so that K_n, K_n^-, and C_k, have the same number of vertices; see below in Table 15, where the first few values are given.

Table 15

κ in K_n, K_n^-, and C_k with Equal Number of Vertices

$\boldsymbol{\lvert V \rvert}$	$\boldsymbol{\kappa(K_n)}$	$\boldsymbol{\kappa(K_n^-)}$	$\boldsymbol{\kappa(C_k)}$
$\boldsymbol{n = 2^k}$			
2	*1*	*0*	*1*
4	*16*	*8*	*4*
8	*262,144*	*196,608*	*384*
16	*72,057,594,040,000,000*	*63,050,394,780,000,000*	*42,467,330*
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\boldsymbol{n = 2^k}$	$n^{(n-2)}$	$n^{(n-2)} \cdot \left(1 - \frac{2}{n}\right)$	$2^{2^k-k-1} \prod_{i=1}^{k} i^{\binom{k}{i}}$

Interestingly enough, the *n*-cube graph family appears to have the least number of spanning trees when comparing graphs with the same $|V|$. But recall that when n is used as an index, the number of spanning trees for the *n*-cube graph family is large compared to the other graph families. It makes sense that n-cube graph family has smaller κ because C_k has $\frac{kn}{2}$ edges, which is far fewer than $\frac{n(n-1)}{2}$ in K_n (or even $\frac{n(n-1)}{2} - 1$ in K_n^-).

Next, we compare B_n and O_n to C_n since both B_n and O_n have $2n$ vertices, Table 16 contains κ when these families have the same $|V|$.

Table 16

κ in O_n, B_n, and C_n with Equal Number of Vertices

$\boldsymbol{\lvert V \rvert}$	$\boldsymbol{\kappa(O_n)}$	$\boldsymbol{\kappa(B_n)}$	$\boldsymbol{\kappa(C_{k+1})}$
$\boldsymbol{2n = 2(2^k)}$			
2	0	0	1
4	4	0	4
8	82,944	384	384
16	24,759,630,000,000,000	513,684,800,000	42,467,330
32	5.08206×10^{44}	1.681479×10^{35}	2.077602×10^{19}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\boldsymbol{2n = 2(2^k)}$	$n^{n-2} \cdot 2^{2n-2} \cdot (n-1)^n$	$(n-1) \cdot n^{n-2} \cdot (n-2)^{n-1}$	$2^{2^k-k-1} \prod_{i=1}^{k} i^{\binom{k}{i}}$

The same idea occurs in this set of graph families; the number of spanning trees increases faster for B_n and O_n than for C_n when the graph families have the same number of vertices. This also is likely because the number of edges for C_{k+1} is kn, which is fewer than $n^2 - n$ edges in B_n and $2n(n-1)$ edges in O_n. This suggests the following for future research.

Conjecture 4.5: The graph C_n has the least number of spanning trees when the number of vertices is fixed compared to the other graph families K_n, K_n^-, B_n and O_n.

Comparing κ in Graph Families When $|E|$ is the Same

Unlike K_n and K_n^- which have the same number of vertices n, or O_n and B_n which both have $2n$ vertices, the formula for the number of edges is different for each graph family. We want to see if there are graphs in the families that have similar numbers of edges and how this affects the number of spanning trees. In Table 17 we can see edge counts for the first members in each of the graph families.

Table 17

Number of Edges in all Graph Families

$\boldsymbol{n}$	$\boldsymbol{\vert E\vert}$for $\boldsymbol{K_n}$	$\boldsymbol{\vert E\vert}$ for $\boldsymbol{K_n^-}$	$\boldsymbol{\vert E\vert}$ for $\boldsymbol{B_n}$	$\boldsymbol{\vert E\vert}$ for $\boldsymbol{O_n}$	$\boldsymbol{\vert E\vert}$ for $\boldsymbol{C_n}$
1	0	0	0	0	1
2	1	0	2	4	4
3	3	2	6	12	12
4	6	5	12	24	32
5	10	9	20	40	80
6	15	14	30	60	192
7	21	20	42	84	448
8	28	27	56	112	1024
⋮	⋮	⋮	⋮	⋮	⋮
$\boldsymbol{n}$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}-1$	n^2-n	$2n(n-1)$	$n(2^{n-2})$

Dual graphs have the same number of edges. For example, since C_3 and O_3 are dual graphs, they will have the same $|E|$ and κ by Theorem 2.6. And isomorphic graphs, like B_4 and C_3 also have the same $|E|$ and κ. Another example of isomorphic graphs in these graph families are C_2 and O_2; both are the $4-cycle$ graphs. Now let us focus on graphs like C_3, O_3 and B_4 or C_2 and O_2, which have the same $|E|$ and κ.

Question 4.6: In this thesis, we've found examples of graphs with the same $|E|$ and κ because the graphs are isomorphic or dual. Are there examples of graphs with the same $|E|$ and κ other than isomorphic or dual graphs?

As we extended the list of edges per family from the table above, we found a few graphs that had the same number of edges but not the same number of spanning trees. The graphs with the smallest equal $|E|$ that we found was K_4 and B_3, which both have six edges but with 16 and 6 spanning trees, respectively. Another example of many, K_7^- and B_5 have $|E| = 20$, and $\kappa(K_7^-) = 12{,}005$ while $\kappa(B_5) = 40{,}500$. K_{14}^- and B_{10} is another example, which both have $|E| = 90$, and $\kappa(K_{14}^-) = 4.859478 \times 10^{13}$ while $\kappa(B_{10}) = 1.20796 \times 10^{17}$. We also found K_{38}^- and B_{27} both have $|E| = 702$, and $\kappa(K_{38}^-) = 7.058759 \times 10^{56}$ while $\kappa(B_{27}) = 3.511621 \times 10^{73}$. Yet another example is O_{15} and B_{21} which both have $|E| = 420$, and $\kappa(O_{15}) = 2.580252197 \times 10^{65}$ while $\kappa(B_{21}) = 9.960213 \times 10^{51}$. A final example we identified was K_{21} and B_{15} with $|E| = 210$ and $\kappa(K_{21}) = 1.32485 \times 10^{25}$ while $\kappa(B_{15}) = 1.072806 \times 10^{32}$. These are just a few examples of graphs in these families that have common edge counts. However, the κ is not the same for these pairs with the same $|E|$.

Recall that, compared to the other graph families, $\kappa(C_n)$ is much bigger for a fixed n, since the number of vertices is increasing much faster than in the other families. However, $\kappa(C_n)$ is much smaller if the number of vertices is fixed, since the number of edges in C_n does not increase as fast as the other families. This suggests:

Question 4.7: How is κ related to |V| and |E|?

To start thinking about this question, we can compare K_n, B_n, and O_n since B_n has twice the number of edges of K_n and O_n has twice the number of edges of B_n. That is, for K_n, $|E| =$

$\frac{n^2-n}{2}$, for B_n, $|E| = n^2 - n$ and for O_n, $|E| = 2(n^2 - n)$. Table 18 list a few of these values. In one case, K_3 and B_3, both the edge count and κ double from three to six.

Table 18

$|E|$ and κ for K_n, B_n, and O_n

	K_n		B_n		O_n	
n	$\|E\|$	κ	$\|E\|$	κ	$\|E\|$	κ
2	1	1	2	0	4	4
3	3	3	6	6	12	384
4	6	16	12	384	24	82,944
5	10	125	20	40,500	40	32,768,000
6	15	1296	30	6,635,520	60	20,736,000,000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
	$\frac{n^2-n}{2}$	n^{n-2}	n^2-n	$(n-1)\cdot n^{n-2}\cdot(n-2)^{n-1}$	$2(n^2-n)$	$n^{n-2}\cdot 2^{2n-2}\cdot(n-1)^n$

Question 4.8: Does having twice the number of edges yield some relationship in the number of spanning trees in these graph families?

These questions above reminded us of the Euler characteristic; thank you to Dr. Vallieres for suggestion this connection. The Euler characteristics of a graph G is $|V| - |E|$.

Question 4.9: Can the Euler Characteristic help us find graphs that have the same κ?

While comparing κ to $|V|$ in these five graph families resulted in a conjecture, the relationship between κ and $|E|$ only suggested questions. Next, we use the O_n graph family to suggest a conjecture and a question.

$\kappa(K_{2n}/k\ edges)$ for $k > 1$

As we looked at our Octahedral Graph Family, we noticed that it was equivalent to K_{2n} / n edges and we decided to expand on this idea to form another future research question. As we mentioned in Chapter 3, the graph family K_{2n} / n edges is the same as O_n, as we noticed after calculating $\kappa(O_n)$. For K_{2n} / n edges to be isomorphic to O_n we must remove the n edges in such a way that we have $deg(v) = 2(n-1)$ for all vertices of the graph K_{2n} / n edges. This means that no pair of removed edges share a vertex.

By Cayley's Formula $\kappa(K_{2n}) = (2n)^{2n-2}$ and by Theorem 2.3 we have $\kappa(K_{2n}^-) = (2n)^{2n-2}\left(1 - \frac{2}{2n}\right)$. It is striking that for K_{2n} / n edges we have

$$\kappa(K_{2n}/n\ edges) = \kappa(O_n) = (2)^{2n-2}(n)^{n-2}(n-1)^n$$

$$= (2)^{2n-2}(n)^{2n-2}(n)^{-n}(n-1)^n$$

$$= (2)^{2n-2}(n)^{2n-2}\left(\frac{n-1}{n}\right)^n$$

$$= (2)^{2n-2}(n)^{2n-2}\left(1 - \frac{1}{n}\right)^n$$

Thus $\kappa(K_{2n}/n\ edges) = (2n)^{2n-2}\left(1 - \frac{2}{2n}\right)^n$, which is similar to the formulas for $\kappa(K_{2n}) = (2n)^{2n-2}$ and $\kappa(K_{2n}^-) = (2n)^{2n-2}\left(1 - \frac{2}{2n}\right)$. This suggest the following conjecture.

Conjecture 4.10: Let $n \geq 1$. If, for $0 \leq k \leq n$, k edges are removed from K_{2n} so that no pair of edges share a vertex, then $\kappa(K_{2n}/k\ edges) = (2n)^{2n-2}\left(1 - \frac{2}{2n}\right)^{k}$.

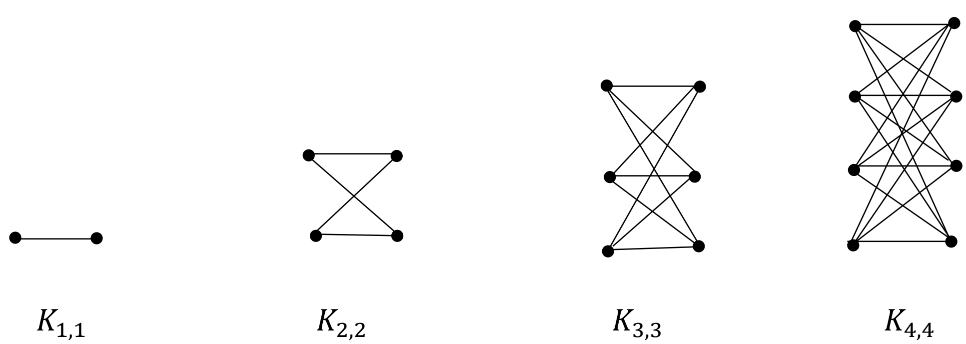
In this thesis we have seen that the conjecture holds when $k = 0, 1$, or n. When $k = 0$, $\kappa(K_{2n}/0\ edges) = (2n)^{2n-2}\left(1 - \frac{2}{2n}\right)^{0}$ is true by Cayley's Formula: $\kappa(K_{2n}) = (2n)^{2n-2}$. When $k = 1$, $\kappa(K_{2n}/1\ edges) = (2n)^{2n-2}\left(1 - \frac{2}{2n}\right)^{1}$ is true by Theorem 2.3. Note that unless $k = 0$ or *n,* these graphs are not regular, and we cannot apply Corollary 2.7.

Graph Family $K_{n,n}$ and $K_{n,n}^{-}$

Since we found similarities between κ for K_{2n}, K_{2n}^{-}, $K_{2n}\backslash n\ edges$ and possibly $K_{2n}\backslash k\ edges$ we decided to look at $K_{n,n}^{-}$ and find a formula for its spanning trees to find similarities in κ for $K_{n,n}$, $K_{n,n}^{-}$ and B_n. We will discuss the complete bipartite graph where *m* and *n* are the same. Figure 13 has the first four complete bipartite graphs, $K_{n,n}$.

Figure 13

Complete Bipartite Graphs



A spanning tree in $K_{n,n}$ will have $2n - 1$ edges and we know that $K_{n,n}$ is n −regular. By the Degree Theorem $K_{n,n}$ has $\frac{2nn}{2} = n^2$ edges. Scoins' Formula provides the number of spanning trees for a complete bipartite graph: $\kappa(K_{n,m}) = n^{m-1}m^{n-1}$ (Scoins 1962). For our graph we have $n = m$, so $\kappa(K_{n,n}) = n^{n-1}n^{n-1} = n^{2(n-1)}$. The Table 19 provides the number of spanning trees of $K_{n,n}$.

Table 19

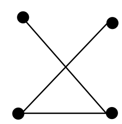
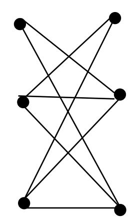
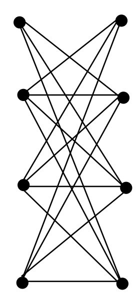
κ for $K_{n,n}$

Complete Bipartite Graph	$\boldsymbol{\kappa}$
$\boldsymbol{K_{1,1}}$	1
$\boldsymbol{K_{2,2}}$	4
$\boldsymbol{K_{3,3}}$	81
$\boldsymbol{K_{4,4}}$	4,096
$\boldsymbol{K_{5,5}}$	390,625
$\vdots$	$\vdots$
$\boldsymbol{K_{n,n}}$	n^{2n-2}

The number of spanning trees for $K_{n,n}$ increases as the number of vertices increases. Now let us look at $K_{n,n}^-$. Table 20 lists κ for $K_{n,n}^-$

Table 20

Graph and κ for $K_{n,n}^{-}$

Complete Graph minus an Edge	Graph	κ
$K_{1,1}^{-}$		0
$K_{2,2}^{-}$		1
$K_{3,3}^{-}$		36
$K_{4,4}^{-}$		2304
$\vdots$	$\vdots$	$\vdots$
$K_{n,n}^{-}$		$n^{2(n-2)}(n-1)^2$

According to the *Tree Theorem*, a spanning tree in $K_{n,n}$ will have $2n-1$ edges. And by the *Degree Theorem,* since $K_{n,n}$ is $n-regular$, we can find the number of edges in $K_{n,n}$, as $\frac{2n \cdot n}{2} = n^2$. Before proving the general result, let us look at some examples. $K_{1,1}$ has one edge, so

$\kappa(K_{1,1}) = 1$ since it is a tree. From Table 20 we can see that $K_{1,1}^{-}$ is a disconnected graph, hence $\kappa(K_{1,1}^{-}) = 0$. $K_{2,2}$ has 4 vertices, so each spanning tree will have 3 edges by the Tree Theorem. Note that $K_{2,2}^{-}$ will have 3 edges. Since it is a tree, $\kappa(K_{2,2}^{-}) = 1$. Using Scion's formula $\kappa(K_{3,3}) = 81$, and each of its spanning trees has 5 edges by the Tree Theorem since it has 6 vertices. Every edge in a spanning tree of $K_{3,3}$ has a $\frac{5}{9}$ chance of being part of a given tree since $K_{3,3}$ contains 9 edges total. So, in $\kappa(K_{3,3}^{-})$, $1 - \frac{5}{9} = \frac{4}{9}$ of the trees remain. Therefore $\kappa(K_{3,3}^{-}) = 81\left(\frac{4}{9}\right) = 36$. Finally, $K_{4,4}$ has 16 edges, 8 vertices and each spanning tree contains 7 edges. The probability that a removed edge is in a particular tree is $\frac{7}{16}$ and the count of the trees that do not include the removed edge for $K_{4,4}^{-}$ will be $\frac{9}{16}$ of $\kappa(K_{4,4})$. Therefore $\kappa(K_{4,4}^{-}) = 4096\left(\frac{9}{16}\right) = 2304$. Considering the pattern, this suggest that $\kappa\left(K_{n,n}^{-}\right)$ is $K_{n,n}$ times the quantity of one minus the probability of an edge being part of a spanning tree.

Theorem 4.11: For $n \geq 1$, $\kappa(K_{n,n}^{-}) = n^{2(n-2)}(n-1)^2$.

Note that,

$$n^{2n-2} \cdot \left(1 - \frac{2n-1}{n^2}\right) = n^{2n-4} \cdot (n^2 - 2n + 1) = n^{2n-4} \cdot (n-1)^2$$

Proof:

In $K_{n,n}$ there are n^{2n-2} spanning trees and each has $2n-1$ edges. This gives $(n^{2n-2})(2n-1)$ positions for a particular edge to occupy in a spanning tree of $K_{n,n}$. By symmetry each edge will occur in the same number of spanning trees. There are $\frac{2nn}{2} = n^2$ edges in $K_{n,n}$ and each is in $\frac{(n^{2n-2})(2n-1)}{n^2} = (n^{2n-2})(n^{-2})(2n-1)$ trees. When an edge is deleted, these trees are no longer spanning trees of $K_{n,n}^{-}$ and there remain $n^{2n-2} - (n^{2n-2})(n^{-2})(2n - 1) = n^{2n-2}(1 - (2n-1)n^{-2})$ spanning trees which is the same as $n^{2n-4} \cdot (n-1)^2$ as needed.

■

Let us compare κ for $K_{n,n}$, $K_{n,n}^{-}$ and B_n. In Table 21 we can see a pattern in the formulas of these κ.

Table 21

κ for $K_{n,n}$, $K_{n,n}^{-}$ and B_n

Graph Family	κ
Complete Bipartite	n^{2n-2}
Complete Bipartite minus an edge	$n^{2n-4}(n-1)^2$
Bipartite family graph B_n	$n^{n-2}(n-1)^1(n-2)^{n-1}$
Pattern observed	$n^a(n-1)^b(n-2)^c$ where $a+b+c=2n-2$

For the complete bipartite graph, we have $n^a(n-1)^b(n-2)^c$ where $a = 2n-2, b = 0$ and $c = 0$. For the complete bipartite minus an edge graph we have $n^a(n-1)^b(n-2)^c$ where $a = 2n-4, b = 2$ and $c = 0$. And for the bipartite graph B_n we have $n^a(n-1)^b(n-2)^c$ where $a = n-2, b = 1$ and $c = n-1$. For all these we can see that $a+b+c = 2n-2$.

Question 4.12: Let $n \geq 1$. If for $0 \leq k \leq n$, we remove k edges from $K_{n,n}$, will there always be a choice of $a, b,$ and c so that $\kappa(K_{n,n} \backslash k\ edges) = n^a(n-1)^b(n-2)^c$ where $a+b+c = 2n-2$?

We know this is true when $k = 0, 1$ and n. When $k = 0$, then $\kappa(K_{n,n} \backslash 0\ edges) = n^a(n-1)^b(n-2)^c$ where $a = 2n-2, b = 0$ and $c = 0$. When $k = 1$, then $\kappa(K_{n,n} \backslash 1\ edges) = n^a(n-1)^b(n-2)^c$ where $a = 2n-4, b = 2$ and $c = 0$. When $k = n$, then $\kappa(K_{n,n} \backslash n\ edges) = n^a(n-1)^b(n-2)^c$ where $a = n-2, b = 1$ and $c = n-1$.

Another future research question that was suggested by Dr. Vallieres during my thesis defense revolves around the formulas for κ for the five graph families. Theorem 3.21 was proven mainly using Cayley's Formula since most κ formulas contain Cayley's Formula except for C_n, refer to Table 9 in Chapter 3.

Question 4.13: Is there a relationship between the graph families, K_n, K_n^-, B_n, and O_n, that causes Cayley's Formula to appear in each formula for κ?

The final question for future research revolves around the idea of non-regular graphs and the Matrix Tree Theorem.

Matrix Tree Theorem for Non-regular Degree Graph Families

In this thesis we have found the number of spanning trees for the new graph families, B_n, and O_n. The graphs, K_n, C_n, B_n, and O_n, are all of regular degree and this is why we used Corollary 2.8 to the Matrix Tree Theorem 2.7. However, the Matrix Tree Theorem holds for any graph, of regular degree or not.

Question 4.14: Can the Matrix Tree Theorem be applied to derive formulas for the number of spanning trees in families as we did for the Octahedral and Bipartite graphs, even when the families include non-regular graphs?

In this chapter we have listed many conjectures and questions for future research. In this thesis we determined the number of spanning trees for two new graph families. We explored known graph families to find relationships with the new graphs. We hope that this will inspire others to pursue the answers to these new questions and conjectures in their own research.

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