INTRINSIC KNOTTING OF BIPARTITE GRAPHS

A Thesis

Presented

to the Faculty of

California State University, Chico

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

in

Mathematics Education

by

Sophy F. Huck

Summer 2010

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ACKNOWLEDGMENTS

This work was made possible in part by a grant from the National Science Foundation. NSF DMS award 0648764 supported a Research Experience for Undergraduates and Teachers (REUT) at California State University, Chico, in the summer of 2007. This paper began with research conducted during that program with Miguel-Angel Manrique and Alexandra Appel, under the supervision of Professor Thomas Mattman. That research will be published in the Kobe Journal of Math [HAMM].

Dr. Mattman has provided me with unquantifiable amounts of instruction, discussion, and motivation. Throughout the research and writing process, Dr. Mattman gave me continued support and feedback, without which I would neither have undertaken nor succeeded with this project. The research of Dr. Mattman and his previous REUT students was instrumental in obtaining the results of this paper.

I would also like to thank the Graduate Coordinator, Dr. Yuichi Handa, and my Thesis Committee members, Drs. Robin Soloway and Benjamin Levitt, for their assistance and insights. Dr. Soloway's careful readings of my thesis provided an invaluable perspective, and resulted in the refinement and clarification of many points.

California State University, Chico has been the home of the bulk of my academic endeavors, from my Bachelor’s Degree to my teaching credential, and now to my Master’s. I could not be more pleased with the education and opportunities that it has provided me.

Finally, I acknowledge my fellow scholar, Michael Ontiveros, for engaging me in mathematical discourse, guiding me through the intricacies of LaTeX, and bolstering my academic self-esteem when I needed it.

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ABSTRACT

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We further identify and categorize intrinsically knotted bipartite graphs. We are motivated by a conjecture that a bipartite graph is intrinsically knotted when the number of edges, E, is greater than or equal to four times the number of vertices, V , minus 17. Previous research by Collins et al. has shown that this is the best possible bound for bipartite graphs that have exactly five vertices in one part and at least five in the other. Our research verifies the conjecture for graphs that have exactly six vertices (respectively exactly seven) in one part and at least six (resp. exactly seven) in the other. We also provide similar bounds for all bipartite graphs.

CHAPTER I

INTRODUCING GRAPHS AND KNOTS

In this chapter we define graphs and bipartite graphs, knots and knotted graphs, and intrinsic knotting. We then present an overview of this thesis.

Within the realms of knot theory and graph theory, students of mathematics have tremendous opportunities to profoundly expand their understanding. The corpus of work on these topics is expanding at a furious rate, but the background needed to contribute original results is entirely learnable. Join me now as we journey through some simple concepts and definitions.

We first must understand what is meant by a graph. In graph theory, a graph consists of a set of vertices, which we can think of as points, and a set of edges, which we can think of as curves joining the vertices. (We'll give a more formal definition soon.) Graph theory is a recent branch of mathematics that can explain and solve many problems both practical and theoretical. The field arose from the mundane situation of taking an evening stroll. In Prussia, on the Pregel river, there were two islands connected to each other and the river banks by seven bridges (see Figure 1). As people crossed the bridges, they wondered, was it possible to walk over each bridge exactly once? This question became known as <u>The Seven Bridges of Königsberg</u> problem. When Leonhard Euler addressed it in a 1741 paper, he essentially created the field of graph theory. Euler saw that the salient aspect of the problem was the connections between the landmasses and the bridges. He replaced each landmass with a vertex, and each bridge with an edge. He then determined that there was no Euler path contained in the graph. That is, one could not devise a walk that would cross each bridge exactly once, and hence, graph theory was born.

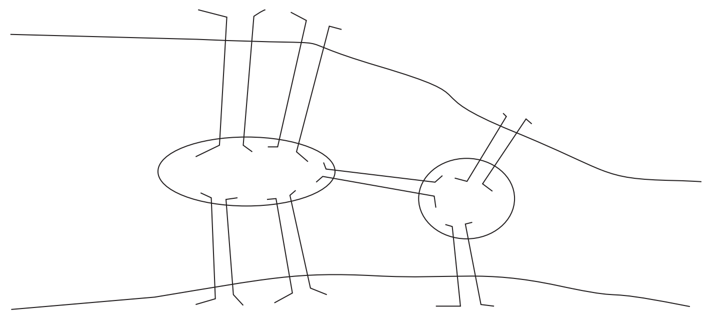


FIGURE 1. The Bridges of Königsberg.

We see applications of graph theory throughout modern life. The World Wide Web is an example of a graph. When you visit a webpage, you are at a vertex. The edges are links that allow you to travel from one page to another. Graph theory informs the methods used by search engines. Social networking sites have members as vertices with each connection between friends as an edge. Of course, networks of relationships between people can be represented by graphs without the context of the internet.

Graph theory pops up all over the place when we look at maps. Figure 2 shows one way that South America could be drawn from the perspective of graph theory. The countries have been replaced by vertices, and edges connect adjacent nations. One can see that Brazil, on the middle right, shares borders with ten other countries. Suppose we want to determine how many different colors would be needed to color the map of South America, with the condition that adjacent countries may not share a color. In graph theory terms, the problem asks how many different colors would be required to color the graph's vertices so that no adjacent vertices share the same color. It was postulated in the mid-nineteenth century that any map (or graph) can be colored this way with four colors. Later that century, mathematicians were able to prove that this can be done with five colors, but the Four Colour Theorem proved trickier. It remained a conjecture until 1976, when it became the first major theorem to be proved by computer (proof by Haken and Appel). Quite a bit of controversy surrounded its proof because the computer algorithms could not be easily reproduced by hand.

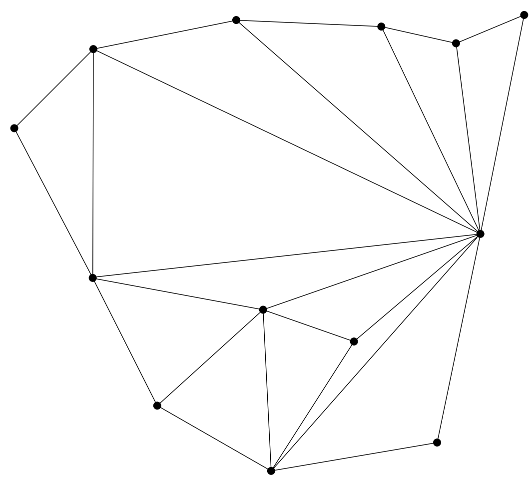


FIGURE 2. South America as a graph

Let's return now to our academic discussion.

Bipartite Graphs

Consider the finite set $V_n = \{a_1, \ldots, a_n\}$ and the limited Cartesian product $E_n = \{(a_i, a_j) : a_i, a_j \in V_n, i \neq j\} \subset V_n \times V_n$. We call elements $a_i \in V_n$ vertices, and elements $(a_i, a_j) \in E_n$ edges where the two edges (a_i, a_j) and (a_j, a_i) are considered the same.

A graph G on n vertices is the union of V_n (the vertices) and some subset of E_n (the edges). In particular, if we represent vertices by points in $\mathbb{R}^3$ and edges by curves joining pairs of vertices, we have what is called a spatial embedding. We require that the curves representing edges can only intersect at their endpoints.

Any particular graph will have many spatial embeddings, since there are many choices for where to place the points and curves. As we define more precisely below, we'll be looking for knots in the various embeddings of a graph.

A complete graph contains all possible edges between vertices. We name such a graph with the notation K_n, where n is the number of vertices in the graph.

Multipartite graphs have two or more disjoint sets of vertices, called the parts, where vertices from one part are connected only to vertices from the other parts. Vertices in the disjoint sets do not share any edges between themselves. The research presented here is specifically concerned with bipartite graphs (i.e., there are two parts, as indicated by the prefix bi).

We refer to G (a bipartite graph) with the following notation:

$$K_{a,b} \setminus me.$$

$K_{a,b}$ is the symbol for a complete bipartite graph, where a and b communicate the number of vertices in each part. A complete bipartite graph includes all possible ab edges. For instance, a complete bipartite graph with two vertices in each part would contain four edges. We will often refer to the two parts as part a and part b.

The symbols a and b communicate how many vertices are in each of the two parts of the graphs. The letter e stands for edges, and is used only when the graph is a certain number of edges, m, short of being complete. Note that $K_{a,b} \setminus me$ actually refers to a collection of graphs. The notation $K_{8,7} \setminus 12e$ refers to a set of graphs each of which has 15 vertices total, eight in one part and seven in the other. There are 12 edges missing compared to the complete graph. As the complete graph has 56 edges, a $K_{8,7} \setminus 12e$ graph will have 44 edges. Notice that there are many ways to remove 12 edges, so this notation does not refer to a single graph. Note also that $K_{a,b}$ and $K_{b,a}$ are two ways of denoting the same graph. We usually write the larger part first.

Figure 3 is an example of the complete bipartite graph $K_{3,3}$ with all nine possible edges present and Figure 4 is an example of an incomplete bipartite graph.

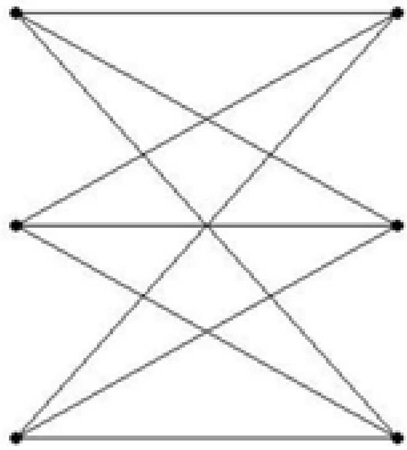


FIGURE 3. $K_{3,3}$.

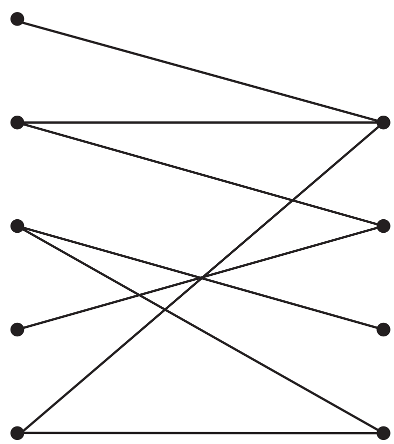


FIGURE 4. An example of a $K_{5,4} \setminus 12e$ graph.

In Figure 4, one part of the graph has five vertices, and the other has four. There are eight edges present. The complete graph would have 20 edges, so we can refer to this graph as a $K_{5,4} \setminus 12e$. Again, this is only one particular graph in the family of $K_{5,4} \setminus 12e$ graphs.

It is simple to draw a graph with only eight edges. However, a graph with more vertices and fewer edges removed would result in a complicated and confusing diagram. A complement graph of a graph G is the graph on the same vertices that show only the edges that are missing in G. A special type of complement graph called a bipartite complement graph shows only the edges that are missing between the two parts in $K_{a,b}$ (since a bipartite graph is by definition missing the edges within parts). So, the bipartite complement of a $K_{a,b} \setminus me$ is also a bipartite graph. It's a $K_{a,b} \setminus (ab - m)e$ graph. For example, a $K_{6,6} \setminus 12e$ would have 24 edges, but is missing only 12 edges between parts. The bipartite complement graph of twelve edges (such as the one in Figure 5) provides a clearer representation of the characteristics of the graph.

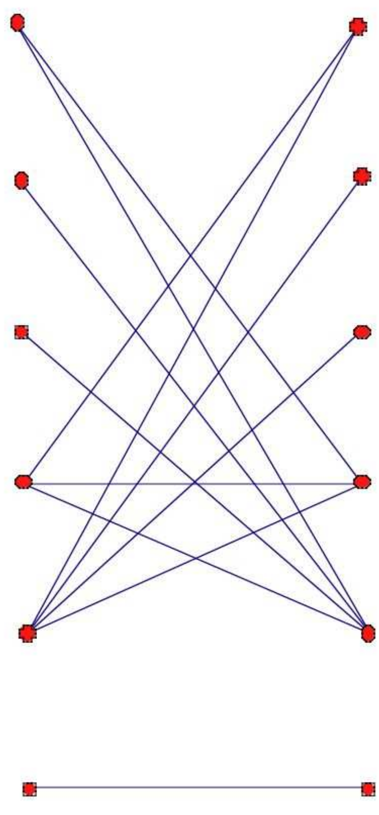
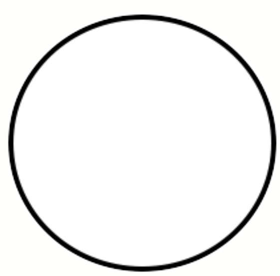


FIGURE 5. This is the bipartite complement of H_{66}, a $K_{6,6} \setminus 12e$ graph.

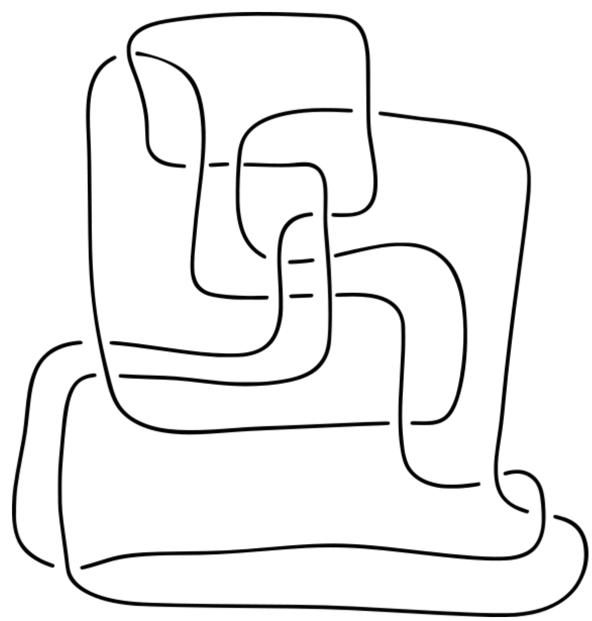
Knots

A <u>knot</u> is a simple closed curve in $\mathbb{R}^3$.

The most basic knot, or trivial knot, is the unknot. The unknot, as in Figure 6, often appears in disguise. However, we can uncover its true identity through the concept of knot equivalency. A knot is a 3-dimensional construct, but it can be represented in 2-dimensional space. A <u>knot diagram</u> is a projection of the knot onto the plane. We will assume the projection is chosen so that at most two points lie on the curve above a given point in the plane (i.e., no triple points). The 3-dimensional crossings of the knot are represented by a broken line that shows which segment is above and which is below.



(a) One representation of the unknot



(b) Another representation of the unknot

FIGURE 6. The unknot.

Notice that Figure 6 consists of two diagrams of the unknot. Two knots are <u>equivalent</u> if a diagram of one can be deformed into a diagram of the other through a series of simple deformations called <u>Reidemeister moves</u>. Figure 7 includes images illustrating the three Reidemeister moves.

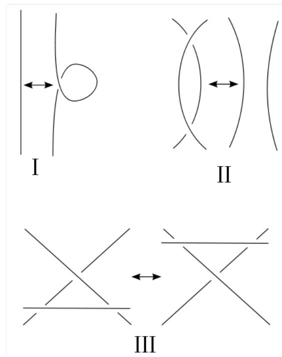


Figure 7.
Illustrations of the
Reidemeister moves.

Using the first Reidemeister move, you may add a twist to your knot in either direction (or remove one). With the second move, you may slide one segment of your knot over another, or vice versa. The third Reidemeister move allows you to slide a segment of the knot past a crossing.

Look at the two representations of the unknot in Figure 8. We may use Reidemeister moves on the right figure to deform it into the left figure. All we would need to do is use the first Reidemeister move twice, and we will be back to our uncomplicated unknot. Notice, we would have to apply many more Reidemeister moves to unravel the complicated unknot in Figure 6.

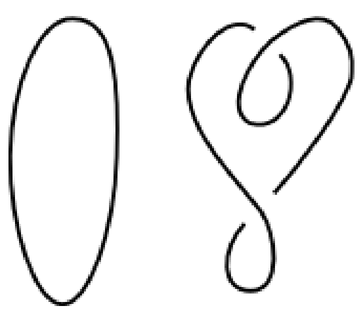


Figure 8. Unknot on
the left; unknot on the
right.

Knots in Bipartite Graphs and
Intrinsic Knottiness

To recognize the presence of a knot in a spatial embedding of a graph, we need to examine the notion of a cycle inside a graph.

Within a representation of any graph, we can travel a path starting from one vertex and ending at another. For example, in Figure 9, we could travel from vertex A to vertex C via vertex B and the two edges that connect those three vertices. A cycle is a path that begins and ends at the same vertex. We can identify a cycle in Figure 9 that begins at vertex B, travels to vertex C, then vertex D, and finally back again to vertex B. Because we both began and ended at vertex B, this path is a cycle. A cycle need not include all vertices and edges of the graph, but it may not repeat edges or vertices.

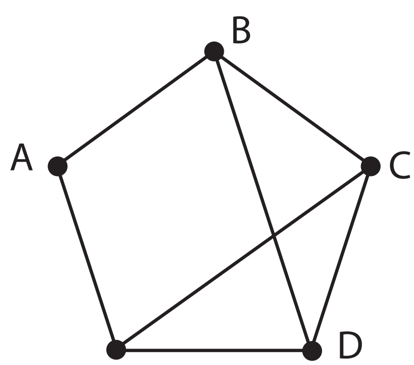


FIGURE 9. A path, ABC, and a cycle, BCDB.

Through examination and proof, we can determine if a particular spatial embedding of a graph contains a knotted cycle. This is a cycle that contains a knot other than the unknot. This can be challenging, because any particular embedding of a graph will have many cycles. Also, as shown by Figure 6, it is not always easy to recognize that a simple closed curve is an unknot.

A graph G is intrinsically knotted, or IK for short, if every spatial embedding of G contains a knotted cycle.

It is important to understand that intrinsic knottiness is a property of the graph, not of the particular embedding or cycle. The goal of our research is to further identify and categorize the existence of intrinsically knotted bipartite graphs.

There are several methods we can use to show that a graph is IK. Our favored method is to show that a graph has an IK subgraph or an IK minor. A subgraph of the graph G (as shown in Figure 10) is a graph obtained from G through a sequence of edge deletions and/or vertex deletions. An edge deletion is the deletion of an edge. A vertex deletion removes a vertex and all of the edges attached to it.

FIGURE 10. The graph at right is a subgraph of the graph at left, obtained through edge and vertex deletions.

Minors are a larger category than subgraphs, as we can in addition contract edges. An edge contraction is when an edge V_1V_2 is removed and the two vertices V_1 and V_2 are combined into one new vertex. This new vertex is adjacent to every vertex that was adjacent to V_1 or V_2. In this paper, we will consider G to be a minor of itself. A vertex split is essentially the opposite of an edge contraction—replace a single vertex, V, with two new vertices, V_1 and V_2, connected by a single (new) edge. Each edge that was connected to V is now connected to either V_1 or V_2. Figure 11 shows an edge contraction from left to right, and a vertex split moving from right to left.

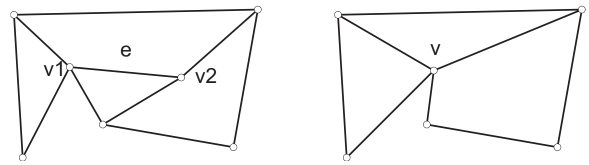


FIGURE 11. An edge contraction (left to right) and a vertex split (right to left).

It's clear that if a subgraph of G is IK, so is G. Although it is not immediately obvious, G is also IK if it has an IK minor [NT]. So, if we show that G has either an IK subgraph or an IK minor, then we will have shown that G is IK.

We will rely on the existence of a class of graphs described in a paper by Kohara and Suzuki [KS]. We will refer to them as the KS graphs. These are fourteen graphs that have been obtained from K_7 (shown in Figure 12).

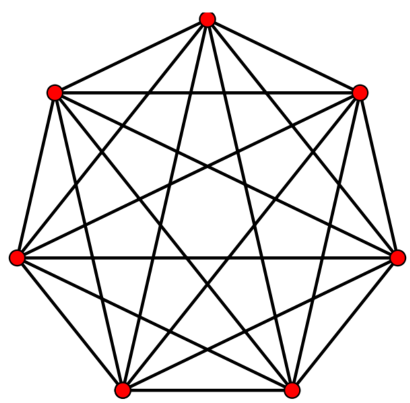


FIGURE 12. K_7 is the basis for the 14 KS graphs.

K_7 is the complete graph on seven vertices—a graph that was shown to be IK by Conway and Gordon [CG]. Recall that since K_7 is a complete graph, it consists of seven vertices, each of which is connected to every other vertex. The KS graphs are obtained from K_7 through a series of ∇Y transformations. This is a process that replaces three vertices, connected as a triangle, with four, connected as a Y shape (Figure 13). One vertex is added, but the number of edges remains the same. Significantly, this transformation is known to preserve the condition of intrinsic knottiness [MRS], so all of the KS graphs obtained from K_7 through ∇Y transformations are also IK. Note that the reverse move, a $Y\nabla$ transformation, does not preserve IK (in general) [FN].

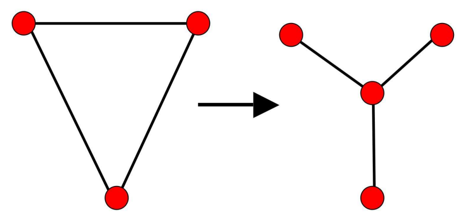


FIGURE 13. ΔY transformations preserve IK.

In particular, we will make use of the KS graphs H_9, F_9, and C_{14} (Figure 14).

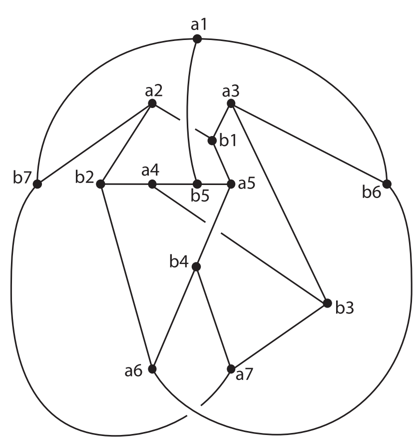
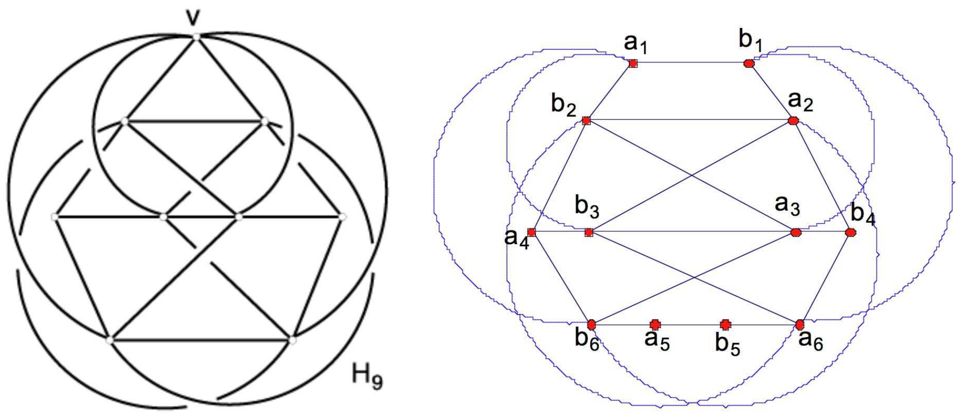


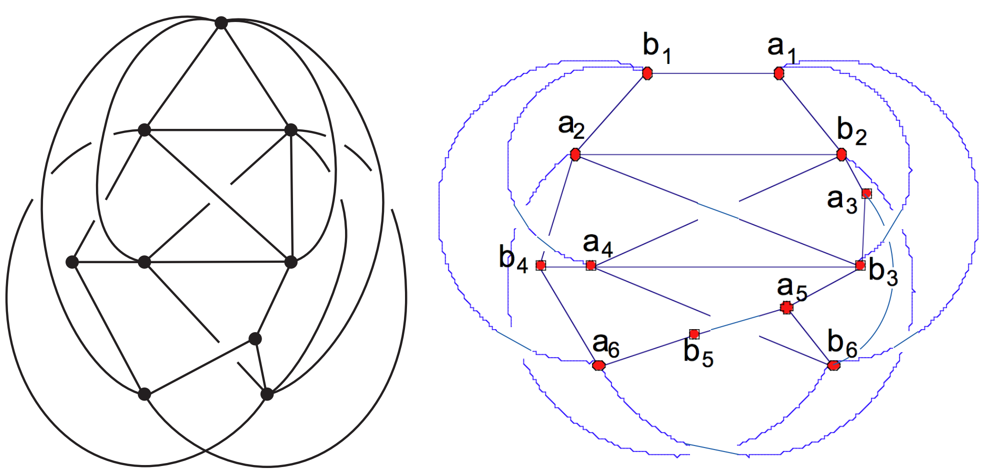
FIGURE 14. C_{14}.

Recall that we can derive an IK graph by splitting vertices of a graph already known to be IK. This method was used to produce H_{66} from the KS graph H_9 (Figure 15), and F_{66} from the KS graph F_9 (Figure 16). Both H_{66} and F_{66} proved useful in determining intrinsic knottiness.



(a) H_9 (b) H_{66}

FIGURE 15. H_9 is a minor of H_{66}; both graphs are IK.



(a) F_9 (b) F_{66}

FIGURE 16. F_9 is a minor of F_{66}; both graphs are IK.

This paper relies on results from Collins, Hake, Petonic, and Sardagna [CHPS]. That paper gives a bound on intrinsic knottiness in bipartite graphs with exactly five vertices in one part and at least five vertices in the other part. They show that such graphs are IK if $E \geq 4V - 17$. Their Corollary 2.5 and Corollary 2.6 prove particularly useful.

Corollary 1 (Corollary 2.5 in [CHPS]):

A graph of the form $K_{a,a+n} \setminus (a+n-3)$ edges is IK for $a \geq 5$, $n \geq 0$.

Corollary 2 (Corollary 2.6 in [CHPS]):

A bipartite graph with 5 vertices in one part, at least 5 vertices in the other, and $E \geq 4V - 17$ is IK.

The salient point of these corollaries is that one can determine intrinsic knottiness by comparing the number of edges in the graph to the number of vertices. It's not necessary to look at every spatial embedding.

This Thesis

This research was motivated by the goal of generalizing the results of [CHPS] by showing that any bipartite graph with V vertices and $E \geq 4V - 17$ edges is intrinsically knotted. This means that if the number of edges is equal to or greater than four times the number of vertices minus 17, then the graph will be IK. Further motivation for the $E \geq 4V - 17$ bound is given in Chapter 2.

This thesis is an extension of work began by this author in two previous papers with Appel and Manrique [HAM] and Appel, Manrique,and Mattman [HAMM]. The work presented here provides a more comprehensive treatment of the material as it fits into the general framework of knot theory and graph theory, including further consideration of the applications of the research. Many of the specific results appear in similar form in the previous papers. The proofs in this paper are often presented with more detail, and sometimes with entirely different methods of proof. These results will be referenced where they a in the text.

In the quest to prove that a bipartite graph with $E \geq 4V - 17$ is intrinsically knotted, several smaller, though enlightening, results came to light.

We first studied some specific cases in an attempt to discover a pattern. We showed that our conjecture is the best possible bound since three of the four $K_{5,5} \setminus 3e$ (for which $E = 4V - 18$) graphs are not IK.

We also found that any graph of the form $K_{6,6} \setminus 5e$ (for which $E = 4V - 17$) is IK, and that a graph of the form $K_{6,6} \setminus 6e$ is IK provided it is not the graph known as G_{666}. This graph can be described with the following notation, which communicates the format of the vertex pairing in this particular case of a $K_{6,6} \setminus 6e$ graph.

$G_{666} = K_{6,6} \setminus \{a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_6b_6\}$, where $a_1 \ldots a_6$ are the vertices in one part, and $b_1 \ldots b_6$ are the vertices in the other part. This notation specifies which six edges have been removed from the graph, as shown in Figure 17.

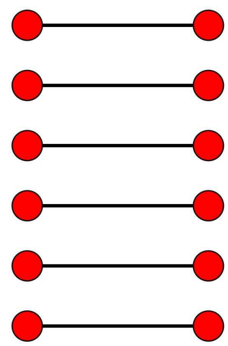


FIGURE 17. The bipartite complement graph of G_{666}.

While we couldn't prove that G_{666} is IK, recent computer experiments by Naimi [N] suggest that it is.

We were also able to show that graphs of the form $K_{7,7} \setminus 10e$ are IK.

The discussion above describes the content of Chapter 2, where we will look at subgraphs of $K_{5,5}$, $K_{6,6}$, $K_{7,7}$, and prove the following theorems:

Theorem 3:

Of the four graphs of the form $K_{5,5} \setminus 3e$, exactly one is IK.

Theorem 4:

Suppose G is a graph of the form $K_{6,6} \setminus 5e$. Then G is IK.

Theorem 5:

A graph of the form $K_{6,6} \setminus 6e$ is IK provided it is not the graph $G_{666} := K_{6,6} \setminus \{a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_6b_6\}$ (see Figure 17).

Theorem 6:

Suppose G is a graph of the form $K_{7,7} \setminus 10e$. Then G is IK.

In Chapter 3, using the results of the previous chapter, we provide a sufficient condition that a subgraph of $K_{a,b}$ be intrinsically knotted for each $a, b \geq 6$. We begin by analyzing graphs with exactly five, six, or seven vertices in one part.

Proposition 7:

Every graph of the form $K_{6+n,6} \setminus (2n+5)e$ is IK where $n \geq 0$.

Proposition 8:

Every graph of the form $K_{7+n,7} \setminus (2n+10)e$ is IK where $n \geq 0$.

Proposition 9:

Every graph of the form $K_{8+n,8} \setminus (2n+15)e$ is IK where $n \geq 1$.

We shall use these to prove the following more general results.

Theorem 10:

Every graph of the form $K_{a,a} \setminus (6a-34)e$ is IK where $a \geq 9$.

Corollary 11:

Every graph of the form $K_{a+n,a} \setminus (3n+6a-34)e$ is IK where $a \geq 9$ and $n \geq 0$.

Finally, in Chapter 4, we summarize our results and suggest questions for additional research.

CHAPTER II

PRELIMINARY LEMMAS AND EXPLICIT EXAMPLES

In this chapter we begin by proving some preliminary lemmas used throughout. We then provide some motivation for the $E \geq 4V - 17$ conjecture, culminating in a proof of Theorem 3 regarding IK for $K_{5,5}$ graphs. We conclude by proving Theorems 4, 5 and 6, which characterize knotting of subgraphs of $K_{6,6}$ and $K_{7,7}$.

Preliminary Lemmas

Recall that if G has an IK subgraph, then G is IK. Accordingly, we find it useful to identify subgraphs of those graphs we wish to prove IK. The following lemmas do just that.

Lemma 12 (Lemma 1 in [HAM]; Lemma 5 in [HAMM]):
Let a, k, and m be positive integers such that $(k-1)(a+1) < m+k$. Then every $K_{a+1,b} \setminus (m+k)e$ graph has a $K_{a,b} \setminus me$ subgraph. In particular, if all graphs of the form $K_{a,b} \setminus me$ are IK, then all $K_{a+1,b} \setminus (m+k)e$ graphs are also IK.

Proof:

Let G be a $K_{a+1,b} \setminus (m+k)e$ graph. If each vertex from the a-part of G is missing less than k edges compared to the complete graph $K_{a+1,b}$, then G is missing at most $(k-1)(a+1)$ edges. Since $(k-1)(a+1) < m+k$ by hypothesis, this contradicts the fact that G is a $K_{a+1,b} \setminus (m+k)e$ graph. Therefore one of the vertices, say a_1, from the a-part of G is missing $k_1 \geq k$ edges. Remove a_1 from G. We are left with a subgraph G' of G of the form $K_{a,b} \setminus (m+k-k_1)e$. Since $k_1 \geq k$, we have that

$m+k-k_1 \leq m$, so that G' (and therefore G) has a $K_{a,b} \setminus me$ subgraph G'' obtained by removing $k_1 - k$ arbitrary edges from G'. The second statement is immediate.

We call a graph G planar if it can be drawn in the plane without any crossings. We denote by K_2+G the union of G and the complete graph K_2 together with all possible edges connecting G and K_2. Note that K_2 is the same graph as $K_{1,1}$.

Lemma 13:

$K_2 + G$ is intrinsically knotted iff G is non-planar.

This lemma is proved in [BBFFHL] and, independently, in [OT].

The proof is based on Kuratowski's Theorem, which states that a graph is planar if and only if it does not contain a K_5 or $K_{3,3}$ minor. Recall that a minor, H', of a graph H is created through a sequence of vertex deletions, edge deletions, or edge contractions.

Proof: sketch $\rightarrow$

We will prove the contrapositive: If G is planar, $K_2 + G$ is not IK. Recall that to be IK, a graph must contain at least one knotted cycle in every spatial embedding. Thus, we need to demonstrate an embedding of $K_2 + G$ that has no knotted cycle. Let's fix an embedding that has G in a plane with two vertices of K_2 on either side of the plane. K_2 adds two connected vertices to G. We begin with G drawn nicely in a plane, as shown in Figure 18.

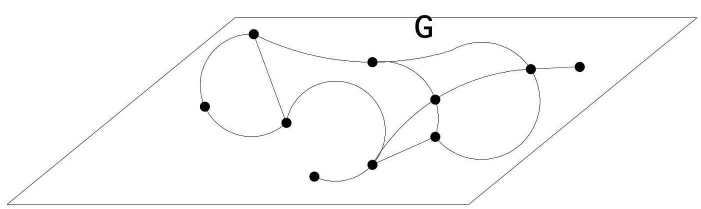


FIGURE 18. An example of the graph G drawn nicely in a plane.

To ensure that the two additional vertices, call them a and b, do not introduce any knots, place a above the plane of G and b below the plane of G. Then connect each a and b to the vertices of G with straight segments, and place the arc connecting a and b well away from the graph of G, as shown in Figure 19.

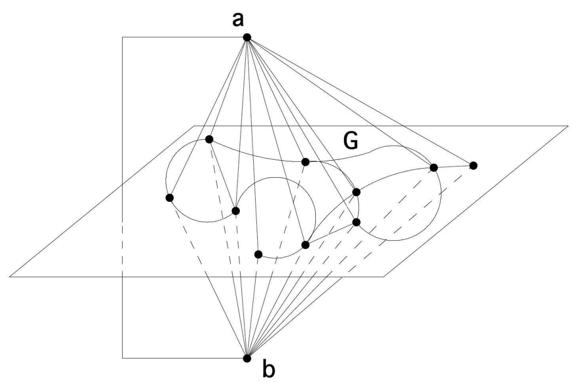


FIGURE 19. A nice embedding of $K_2 + G$.

We can find four different types of cycles in this embedding of $K_2 + G$. If we show that none of these cycles is knotted, then $K_2 + G$ will not be intrinsically knotted.

Case 1: The cycle includes no vertices from K_2 (as shown in Figure 20). This cycle lies completely within the plane of G. Then the cycle lies in a plane and is therefore unknotted.

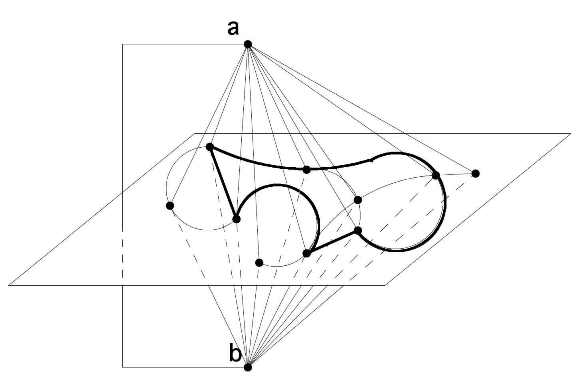


FIGURE 20. Case 1: The cycle is contained in the plane of G.

The following proofs are sketches, as we omit the topological details.

Case 2: The cycle uses one vertex, a, from K_2 (as shown in Figure 21). Since G lies in a plane, this vertex is the only one that lies outside the plane. So, two vertices of G, V_1 and V_2, are connected to a in the cycle. It is intuitve that we can "straighten" the path between V_1 and V_2. Then the whole cycle lies in the triangle $\Delta\ V_1V_2a$. This triangle, in turn, lies in a plane. So, the cycle can be made to lie in a plane (different from the plane of G) and is thus unknotted.

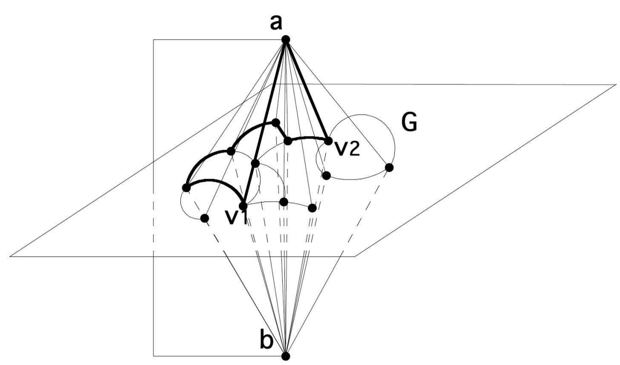


FIGURE 21. Case 2: This "triangular" cycle can be made to lie in a plane.

There are two ways to use both vertices from K_2.

Case 3a: Here the cycle includes segments between a and V_2, V_1 and b, and the arc between a and b (as shown in Figure 22). The cycle also contains a path between V_1 and V_2. As in case 2, "straighten" the path between V_1 and V_2. Then "push" a and b and the edge connecting them into a plane in a "rectangular" cycle. This cycle is unknotted.

Case 3b: Here a is connected to two vertices in G, V_1 and V_4, and b is connected to two vertices in G, V_2 and V_3 (as shown in Figure 23). This cycle does not include the arc between a and b. Again, sketchily "straighten" the path between V_1 and V_3, and the path between V_2 and V_4, and "push" a and b into the plane. This gives a "hexagonal" cycle that lies within a plane.

We've shown that in this embedding, no cycle is knotted.

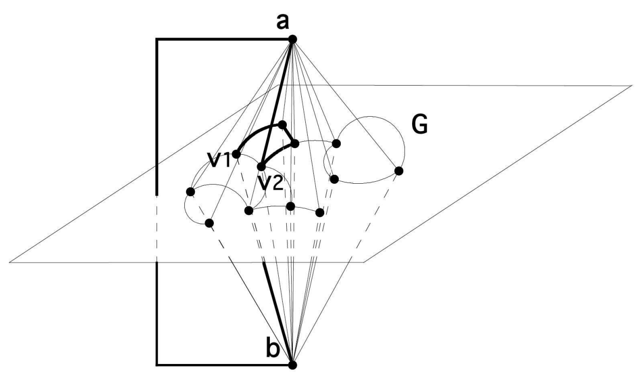


FIGURE 22. Case 3a: This "rectangular" cycle can be made to lie in a plane.

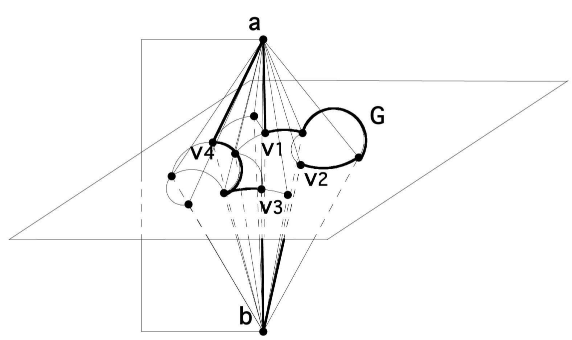


FIGURE 23. Case 3b: This "hexagonal" cycle can be made to lie in a plane.

$\leftarrow$ Suppose G is non-planar. By Kuratowski's Theorem, G then contains a K_5 or $K_{3,3}$ minor. If we add K_2 to K_5, we end up with K_7, which is known to be intrinsically knotted (recall that K_7 is the basis for the KS graphs). If we add K_2 to $K_{3,3}$, we can create a $K_{3,3,1,1}$ quadripartite graph, which was shown to be IK in [F]. Therefore, $K_2 + G$ is intrinsically knotted, as it has either K_7 or $K_{3,3,1,1}$ (both IK) as a minor.

Corollary 14:

If we remove two vertices from a graph H and the resulting graph, G, is planar, then H is not IK.

Proof:

Suppose that G is formed from H by removing two vertices, a and b, and that G is planar. Then H is a subgraph of $K_2 + G$. By Lemma 13, $K_2 + G$ is not IK. Therefore, since H is a subgraph of a non-IK graph, H itself is not IK.

Motivating our Conjecture

In this section we identify examples of intrinsically knotted bipartite graphs for which $E \geq 4V - 17$. We will use Lemma 13 to construct IK graphs from non-planar graphs by adding two vertices. The following two propositions investigate sufficient conditions to produce a non-planar graph. Proposition 15 is a standard result that is included to motivate Proposition 16.

Proposition 15:

If G is a connected graph with $V \geq 3$ vertices and E edges such that $E > 3V - 6$, then G is non-planar.

Proof:

We prove the contrapositive; namely, that if G is planar, then $E \leq 3V - 6$.

The Euler characteristic of any connected planar graph G is $2 = V - E + F$, where V and E are as before, and F is the numbers of faces of a planar representation of G. In a planar embedding of G, let n denote the number of pairs (f, e) of faces f of G and edges e bounding f in G. Since each face f is bounded by at least 3 edges, we have that $n \geq 3F$. Also, since each edge e is shared by at most two faces, we know that $n \leq 2E$. Combining these inequalities yields

$$3F \leq 2E.$$

Plugging into Euler's formula gives $E \leq 3V - 6$.

Therefore by Lemma 13, if a connected graph G has $E > 3V - 6$ edges, then it is non planar, and $K_2 + G$ is IK. Furthermore, we know that $K_2 + G$ has

$E_1 > 5V_1 - 15$ edges, where $V_1 = V + 2$ denotes the number of vertices in $K_2 + G$. We reach the bound of $E > 5V - 15$ through the following calculations:

Begin with G. Adding K_2 gives us

$$E_1 = E + 2V + 1$$

So now, assuming

$$E > 3V - 6$$

$$E_1 > 3V - 6 + 2V + 1$$

$$E_1 \ > 5V - 5$$

and since

$$V = V_1 - 2$$

we have

$$E_1 > 5(V_1 - 2) - 5$$

$$E_1 > 5V - 15$$

We have therefore shown that many graphs with $E > 5V - 15$ are IK. In fact, it can be shown [CMOPRW] that any graph with $V > 6$ vertices and $E > 5V - 15$ edges is IK. Consider, though, a planar graph J that has exactly $3V - 6$ edges. Since J is planar, $K_2 + J$ would not be IK. This tells us that we can not improve on the bound of $E > 5V - 15$ set by [CMOPRW]. For example, consider any graph represented in the plane. If it is not already a triangulation, then it has some faces with more than three edges. Add edges to each of these faces to make them into triangles. Then $E = 3V - 6$. In this way, by starting with arbitrary planar graphs, we can produce an infinite family of non-IK graphs with $E = 5V - 15$.

Next consider a connected planar bipartite graph G. As before, let n denote the number of pairs (f, e) of faces f of G and edges e bounding f in a planar embedding of G. There are no triangles in bipartite graphs, therefore each face f in G is bounded by at least four edges. Accordingly, the inequality becomes $4F \leq 2E$. Plugging back into Euler's formula gives $E \leq 2V - 4$.

We have proved:

Proposition 16:

Suppose $K_{a,b} \setminus me$ is a connected bipartite graph with $E > 2V - 4$ edges. Then $K_{a,b} \setminus me$ is non-planar.

We can use Lemma 13 to construct an IK graph $K_2 + K_{a,b} \setminus me$. $K_2 + K_{a,b} \setminus me$ is IK, but not bipartite. We can, however, use vertex expansion to transform $K_2 + K_{a,b} \setminus me$ into a bipartite graph and thereby retain intrinsic knottiness. Replace each vertex in K_2 with a new pair of vertices (a_1, b_1) and (a_2, b_2). Draw an edge between a_1 and b_1, and a_2 and b_2. Connect both a_1 and a_2 to each vertex on the b-side of $K_{a,b} \setminus me$, and b_1 and b_2 to each vertex on the a-side of $K_{a,b} \setminus me$. Connect a_1 and b_2. This new graph is bipartite and IK, and it has $E > 4V - 17$ edges. To see this, we begin with a graph G that has $E > 2V - 4$ edges. Adding K_2 and expanding its vertices gives us

$$E_1 = E + 2 + 2V + 1$$

So now

$$E_1 > 2V - 4 + 2 + 2V + 1$$

$$E_1 > 4V - 1$$

and since

$$V = V_1 - 4$$

we have

$$E_1 > 4(V_1 - 4) - 1$$

$$E_1 > 4V - 17$$

This is almost the bound of our motivating conjecture, for which $E \geq 4V - 17$.

Note that there's an infinite family of non IK bipartite graphs with $E = 4V - 16$: the $K_{4,n}$ graphs. If we remove two vertices from the part of the graph with four vertices, we obtain a $K_{2,n}$ graph. We can find a planar embedding of this graph by placing the n vertices in a line, and connecting each of them to each of the two vertices in the other part. So $K_{4,n}$ is not IK by Corollary 14. These graphs have $E = 4n = 4(n+4) - 16 = 4V - 16$ edges. We can look at this class of graphs as having $4V - c$ edges where c is a constant. Since we are able to identify many examples of graphs with $E = 4V - c$ that are not IK, this suggests that a bound of the form $E > 4V - c$ is likely the best we can hope for. Note that these examples don't directly contradict our motivating conjecture as they have only four vertices in one part.

In order to determine the correct constant c in the above bound, we need to examine explicit examples. As $K_{a,b}$ is not IK unless a and $b \geq 5$, we start with $K_{5,5}$. Campbell et al. [CMOPRW] have shown that all $K_{5,5} \setminus 2e$ graphs are IK. There are two ways to remove two edges from a $K_{5,5}$ graph. Either the two edges share a vertex, or they don't. Either way, we can show by Figure 24 (from [MOR]) that $K_{5,5} \setminus 2e$ has H_9 as a minor. Recall that H_9 is one of the KS graphs. It is derived from K_7 by 2 ∇Y moves and is therefore IK [KS].

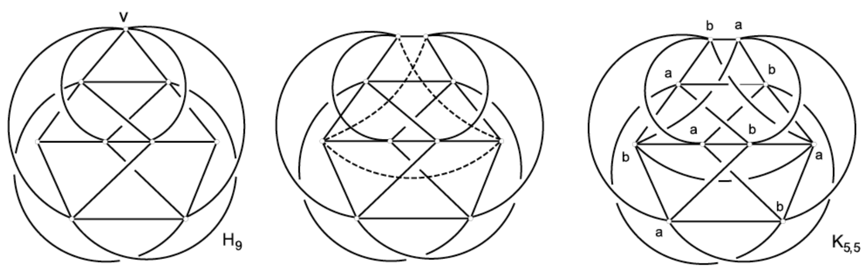


FIGURE 24. H_9 is a minor of $K_{5,5}$. T.W. Mattman, R. Ottman, and M. Rodrigues, 'Intrinsic Knotting and Linking of Almost Complete Partite Graphs,' (preprint). ArXiv:math/0312176v2 [math.GT]. Reprinted with permission.

Since we now know that all $K_{5,5} \setminus 2e$ graphs are IK, we naturally consider the $K_{5,5} \setminus 3e$ graphs. That is, if we remove three edges from a $K_{5,5}$ graph, will the resulting graph still be IK?

There are four $K_{5,5} \setminus 3e$ graphs. It was shown in [CHPS] that one of them is not IK. Below, we show further that three of the four $K_{5,5} \setminus 3e$ (these have 22 edges, so $E = 4V - 18$) graphs are not IK. Therefore, our conjectured bound, $E \geq 4V - 17$ (which has at least one fewer edge removed) is the best possible.

We classified the $K_{5,5} \setminus 3e$ graphs through simple examination.

Theorem 3 (Theorem 1 in [HAM]): Of the four graphs of the form $K_{5,5} \setminus 3e$, exactly one is IK.

Proof:

There are four different ways to remove three edges from $K_{5,5}$. The four respective bipartite complement graphs are shown in Figure 25.

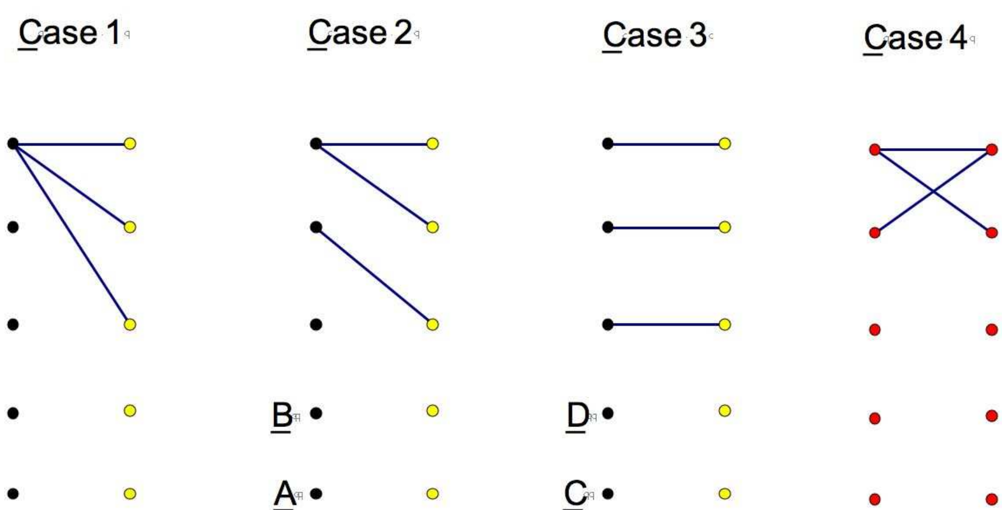


FIGURE 25. The bipartite complement graphs for the four cases of $K_{5,5} \setminus 3e$.

In [CHPS] it was shown that the graph of Case 1 is not IK. Here's the argument. In this case, the three missing edges originate from the same vertex, b_1, in part b, as shown in Figure 25. Let $G = K_{4,4,1} \setminus 2e$, where the two missing edges are between two vertices in part a and the single vertex of the third part, c. This graph was shown to be non-IK in [CMOPRW]. Let $H = K_{4,4,1} \setminus 3e$, where the three missing edges are again between the single vertex of part c and three of the vertices in part a (see Figure 26). Now, H is a subgraph of G obtained by an edge deletion. So, since G is not IK, it follows that its subgraph, H, is also not IK. Add a degree two vertex to the edge between the vertex of part c and the vertex of part a that is not missing an edge. We can then reorganize the vertices to obtain a bipartite graph (where part c has joined part a, and our added vertex has joined part b) missing three edges, all orginating from the same vertex, d. This graph, like H, is not IK.

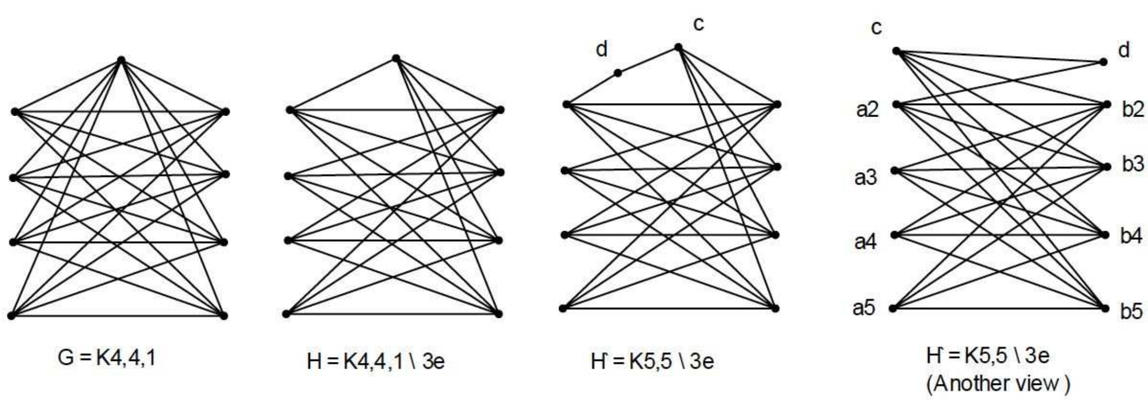


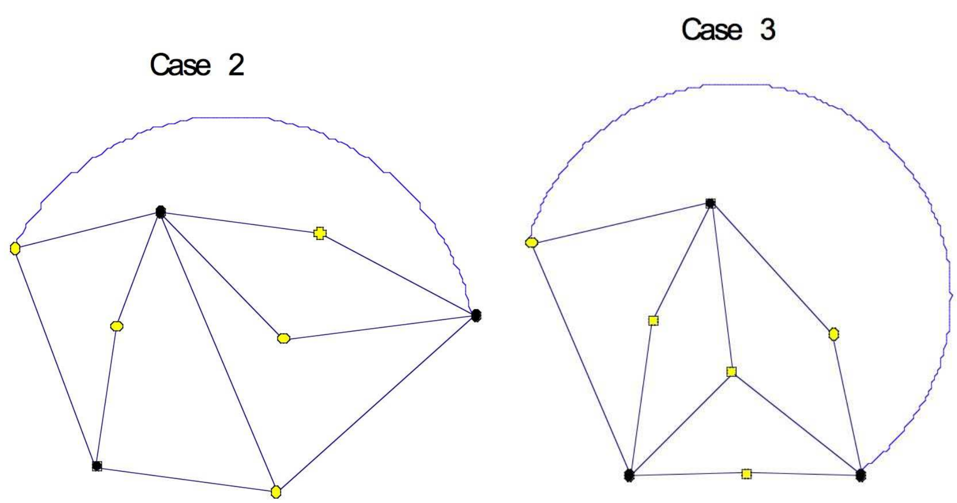
FIGURE 26. Case 1 of $K_{5,5} \setminus 3e$ is not IK.

Since this graph has an unknotted embedding, we now know that the class of graphs $K_{5,5} \setminus 3e$, as a whole, is not IK. However, we are still interested in seeing if we can find a particular graph of this class that is IK. So we will look at the other three cases.

We can show that Cases 2 and 3 are not IK by Corollary 14. According to this corollary, if we take away two vertices, and are able to draw the remaining graph in a plane (as in Figure 27), then the original graph is not IK. The vertices that will be removed are labeled A - D in Figure 25 and will not be shown in the planar graphs in Figure 27. Please note that these graphs are <u>not</u> complement graphs.

Case 4 is the only $K_{5,5} \setminus 3e$ graph that is IK. It has the KS graph H_9 as a minor, as shown in [MOR] (see Figure 24).

We know that $K_{5,5} \setminus 3e$ graphs have $E = 4V - 18$ edges. To count for you, there are ten vertices, four times ten is 40, and 40 - 18 is 22. Another way of looking at this is that a $K_{5,5}$ graph has 25 edges, so a graph of the form $K_{5,5} \setminus 3e$ will have 22 edges. So we know that the correct constant term in our lower bound is less than 18. That is, since the class of $K_{5,5} \setminus 3e$ graphs are not all IK, and they have $4V - 18$ edges, then the best possible bound that we could set is $E = 4V - 17$ edges. This is our conjecture.



(a) (b)

FIGURE 27. In cases 2 and 3, after removing two vertices, we have a planar graph.

Supporting our Conjecture

It is shown in [CHPS] (see Corollary 2 above) that our conjecture holds for graphs of the form $K_{5,n} \setminus me$ when $n \geq 5$. As an example, recall the intrinsic knottiness of $K_{5,5} \setminus 2e$.

We have yet to prove our conjecture in full generality. We can, however, verify it for particular classes of graphs.

Theorem 4 considers the class of graphs $K_{6,6} \setminus 5e$. In terms of our conjecture, this class of graphs meets the condition that $E \geq 4V - 17$. To count for you, four times twelve is 48, and 48 - 17 is 31. This is equal to the number of edges in $K_{6,6} \setminus 5e$, six times six minus five. Indeed, $E = 4V - 17$ for these graphs.

Theorem 4 (Theorem 2 in [HAM]; Theorem 8 in [HAMM]): Every graph of the form $K_{6,6} \setminus 5e$ is IK.

Proof:

Suppose G is formed from the complete bipartite graph $K_{6,6}$ by removing five edges. If there is a vertex a of G that has two or more edges removed compared to $K_{6,6}$, delete it. This leaves a graph of the form $K_{6,5} \setminus me$ with $m \leq 3$. By Corollary 2, this graph is IK. So in this case G has an IK subgraph and thus is IK.

Therefore, we may assume that each vertex of G has at most one edge removed compared to the complete graph $K_{6,6}$. In edgier notation, we can represent this graph as $K_{6,6} \setminus \{a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5\}$. This $K_{6,6} \setminus 5e$ graph has F_{66} (see Figures 28 and 29) as a subgraph and is therefore IK. We see this by noting that the bipartite complement of this graph is a subgraph of the bipartite complement of F_{66}, as in Figure 30. Note that the vertex labeling in this figure refers to the vertices of our graph G.

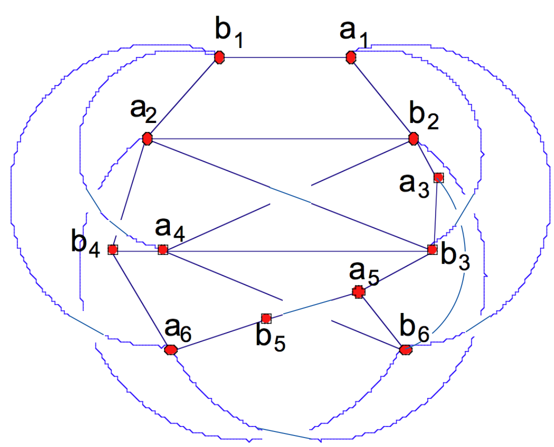


FIGURE 28. F_{66}.

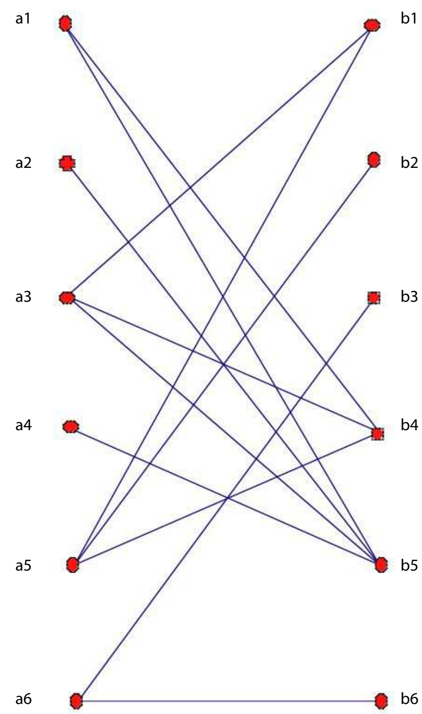


FIGURE 29. The bipartite complement graph of F_{66}.

Theorem 5 (Theorem 3 in [HAM]; Theorem 9 in [HAMM]): A graph of the form $K_{6,6} \backslash 6e$ is IK provided it is not the graph $K_{6,6} \backslash \{a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_6b_6\}$.

Recall that this graph has been given a special name, G_{666}, to specify that it is the one graph of the form $K_{6,6} \setminus 6e$ which no one has either proved or disproved to be IK. The notation used in the theorem indicates that this is the graph where each vertex in part a is missing one edge, and each vertex in part b is missing one edge (see Figure 17).

Proof:

Let H_{66} be the graph of the form $K_{6,6} \setminus 12e$ that has H_9 as a minor shown in Figure 31. Let F_{66} be the graph of the form $K_{6,6} \setminus 12e$ that has F_9 as a minor shown in Figure 28. Our strategy is to show that the $K_{6,6} \setminus 6e$ graphs are IK by demonstrating that they have as a subgraph either H_{66} or F_{66}. Recall that H_{66} and F_{66} are IK, since they have the IK KS graphs H_9 and F_9 (respectively), as minors.

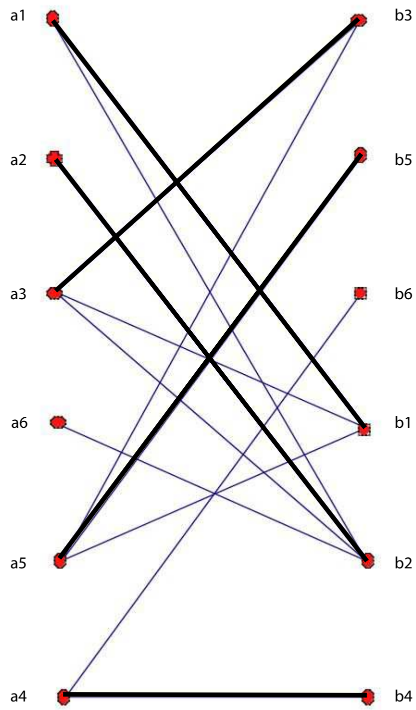


FIGURE 30. The bolded edges show that the bipartite complement graph of this $K_{6,6} \setminus 5e$ is a subgraph of the bipartite complement graph of F_{66}.

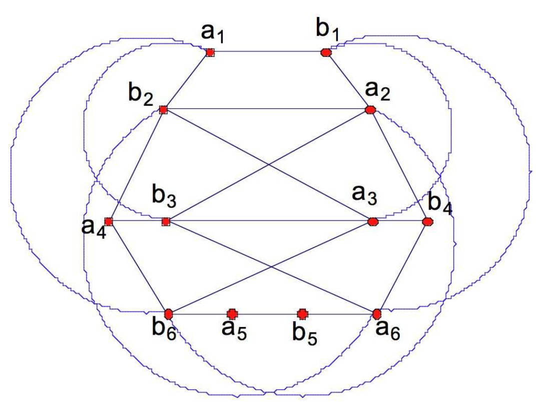


FIGURE 31. H_{66}.

To determine all possible ways to remove six edges from $K_{6,6}$, first let $a_1, \ldots, a_6$ be the vertices in one part and $b_1, \ldots, b_6$ be the vertices in the other part of the graph. Now consider any partition of six and take the i^{th} element in a partition to be the number of edges removed from the i^{th} vertex in the first part of $K_{6,6}$. Likewise, consider another partition and allow its entries to correspond to the number of edges removed from the other part of $K_{6,6}$.

We considered partitions of six to find all ways to remove six edges from $K_{6,6}$, under the condition that no vertex has more than two edges removed. Without this condition, we can remove such a vertex, leaving us with a $K_{6,5}$ graph missing three or fewer edges. By Corollary 1, graphs of this form are IK, so we do not need to consider them further. We found 17 cases:

(i) $K_{6,6} \setminus \{a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_6b_6\}$; $\{\{1,1,1,1,1,1\},\{1,1,1,1,1,1\}\}$

(ii) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_3, a_2b_4, a_3b_5, a_3b_6\}$; $\{\{2,2,2\},\{1,1,1,1,1,1\}\}$

(iii) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_3, a_3b_4, a_4b_5, a_5b_6\}$; $\{\{2,1,1,1,1\},\{1,1,1,1,1,1\}\}$,

$\sigma = (534162, 235146)$

(iv) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_3, a_2b_4, a_3b_5, a_4b_6\}$; $\{\{2,2,1,1\},\{1,1,1,1,1,1\}\}$,

$\sigma = (453612, 142356)$

(v) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_3, a_3b_3, a_4b_4, a_5b_5\}$; $\{\{2,1,1,1,1\},\{1,1,2,1,1\}\}$,

$\sigma = (512463, 125463)$

(vi) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_2, a_3b_3, a_4b_4, a_5b_5\}$; $\{\{2,1,1,1,1\},\{1,2,1,1,1\}\}$,

$\sigma = (541263, 214563)$

(vii) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_2, a_2b_3, a_3b_4, a_4b_5\}$; $\{\{2,2,1,1\},\{1,2,1,1,1\}\}$,

$\sigma = (543612, 241563)$

(viii) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_2, a_3b_3, a_3b_4, a_4b_5\}$; $\{\{2,1,2,1\},\{1,2,1,1,1\}\}$,

$\sigma = (514623, 241563)$

(ix) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_3, a_3b_3, a_4b_4, a_4b_5\}$; $\{\{2,1,1,2\},\{1,1,2,1,1\}\}$,

$\sigma = (423516, 145236)$

(x) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_2, a_2b_3, a_3b_4, a_3b_5\}$; $\{\{2,2,2\},\{1,2,1,1,1\}\}$,

$\sigma = (415236, 145236)$

(xi) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_2, a_2b_3, a_3b_4, a_4b_4\}$; $\{\{2,2,1,1\},\{1,2,1,2\}\}$,

$\sigma = (452316, 143526)$

(xii) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_2, a_3b_3, a_3b_4, a_4b_4\}$; $\{\{2,1,2,1\},\{1,2,1,2\}\}$,

$\sigma = (425136, 152436)$

(xiii) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_2, a_2b_3, a_3b_3, a_4b_4\}$; $\{\{2,2,1,1\},\{1,2,2,1\}\}$,

$\sigma = (541632, 215634)$

(xiv) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_1, a_2b_2, a_3b_3, a_4b_4\}$; $\{\{2,2,1,1\},\{2,2,1,1\}\}$,

$\sigma = (542613, 145623)$

(xv) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_2, a_2b_3, a_3b_3, a_3b_4\}$; $\{\{2,2,2\},\{1,2,2,1\}\}$,

$\sigma = (541236, 214536)$

(xvi) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_1, a_2b_2, a_3b_3, a_3b_4\}$; $\{\{2,2,2\},\{2,2,1,1\}\}$,

$\sigma = (145236, 452316)$

(xvii) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_1, a_2b_3, a_3b_2, a_3b_3\}$; $\{\{2,2,2\},\{2,2,2\}\}$,

$\sigma = (451236, 154236)$

Each case except the 1^{st} and the 2^{nd} has H_{66} as a subgraph. We write

$$\sigma := (\prod_{i=1}^{6} \sigma_a(i), \prod_{j=1}^{6} \sigma_b(j))$$

for the explicit embedding of the complement graph into the complement of H_{66} defined by the permutation of vertices

$$[\prod_{i=1}^{6} a_i, \prod_{j=1}^{6} b_j] \mapsto [\prod_{i=1}^{6} a_{\sigma_a(i)}, \prod_{j=1}^{6} b_{\sigma_b(j)}],$$

where σ_a and σ_b are permutations on six elements. The 2^{nd} graph has F_{66} as a subgraph. All graphs, then, except the 1^{st}, (G_{666}), are IK.

The second case has the IK graph F_{66} as a subgraph. We determined this by identifying its bipartite complement graph as a subgraph of the bipartite complement graph of F_{66} (see Figure 29). Recall that we can identify this case with the notation described above:

$$K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_3, a_2b_4, a_3b_5, a_3b_6\} \ ; \ \{\{2,2,2\},\{1,1,1,1,1,1\}\}$$

This notation communicates which six edges this graph is missing. In the bipartite complement graph, three vertices on the a side of the graph, a_1, a_2, and a_3, are each connected to two vertices on the b side, respectively b_1 and b_2, b_3 and b_4, and b_5 and b_6. Figure 32 shows how we identified this same vertex pairing within the bipartite complement graph of F_{66}. Note that the vertex labeling corresponds to the notation of this second case, not F_{66}. We have then shown that this case has F_{66} as a subgraph and is therefore IK.

Similarly, all cases after the second have H_{66} as a subgraph. The notation above explains how we mapped the vertices of the bipartite complement graph of each case to the vertices of the bipartite complement graph of H_{66} (see Figure 5). For example, consider the third case (Figure 33):

$$K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_3, a_3b_4, a_4b_5, a_5b_6\} \ ; \ \{\{2,1,1,1,1\},\{1,1,1,1,1,1\}\},$$
$$\sigma = (534162, 235146)$$

The sigma notation indicates how each vertex of the bipartite complement graph of H_{66} is mapped to each vertex of the case under consideration. For example, the vertex-pair identification of this case tells us that vertex a_1 (corresponding to the first position in the sigma notation) of the graph G is not connected to either b_1 or b_2 (corresponding to the first two positions in the second set of the sigma notation). That same relationship exists between the vertex we have labeled as c_5 and the vertices labeled d_2 and d_3. So, the notation above describes

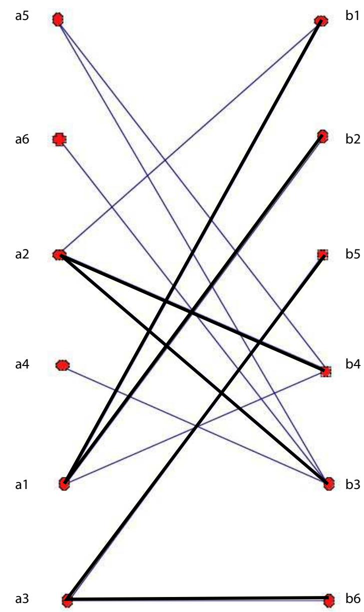


FIGURE 32. The bipartite complement graph of case (ii) (bolded edges) is a subgraph of the bipartite complement graph of F_{66}.

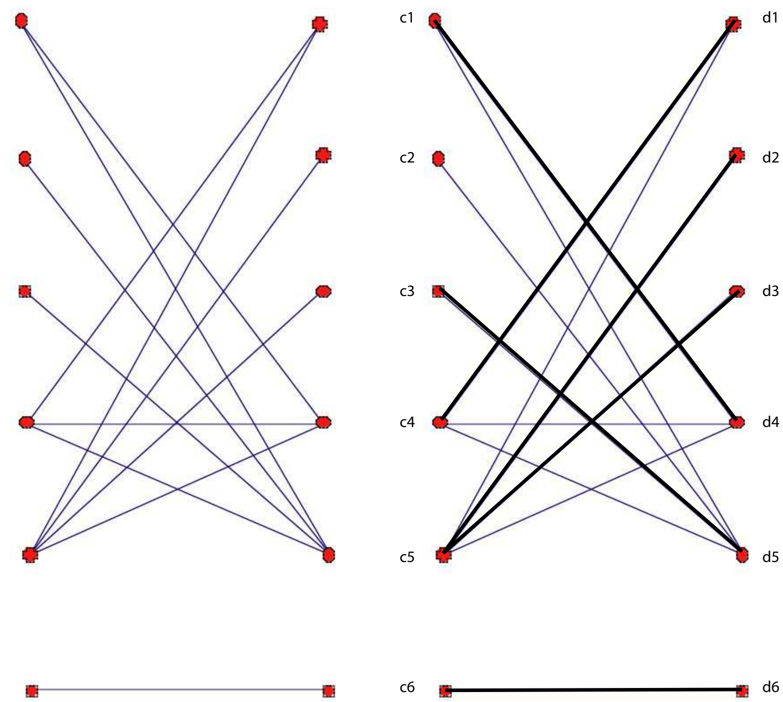
how we mapped the vertex a_1 (b_1 and b_2) of our case to the vertex c_5 (respectively d_2 and d_3) of the bipartite complement graph of H_{66}. We continued in this way to verify that each subsequent case has H_{66} as a subgraph and is therefore IK.

The first graph, $K_{6,6} \setminus \{a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_6b_6\}$ (aka G_{666}), was the only $K_{6,6} \setminus 6e$ graph for which this method was not successful. Recent computer experiments by Ramin Naimi [N] suggest that it is IK.

Theorem 6 (Theorem 4 in [HAM]; Theorem 10 in [HAMM]): Any graph of the form $K_{7,7} \setminus 10e$ is IK.

Proof:

Suppose there exists a vertex with three or more edges missing compared to $K_{7,7}$. Remove it to obtain a $K_{7,6} \setminus me$ subgraph with $m \leq 7$. Propostion 7 below (which



(a) (b)

FIGURE 33. The bolded edges show that the bipartite complement graph of $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_2b_3, a_3b_4, a_4b_5, a_5b_6\}$ is a subgraph of the bipartite complement graph of H_{66}.

does not depend on Theorem 6) shows that such graphs are IK. We can therefore assume that each vertex has at most two edges removed.

We claim that there exist vertices a and b, one in each part, with the property that they are connected by an edge and that each has two edges removed. Since there are ten edges removed, there are at least three vertices on each side with two edges deleted. Call them $a_1, a_2, a_3, b_1, b_2, b_3$. Since we assumed that any a_1 has no more than two edges removed, a_1 must be connected to one of the vertices b_j, to prevent a from having more than two edges removed. So a_1 and b_j are a pair a, b as required. Remove a and b. Now there are four fewer missing edges, so we are left with a $K_{6,6} \setminus 6e$ subgraph. We may assume this graph is G_{666} as otherwise the original graph has an IK subgraph and is therefore IK.

Below we present the three ways to remove two vertices from a $K_{7,7} \setminus 10e$ graph and produce G_{666} (Figure 34). Note that the dark vertices and edges represent the removed vertices and their missing edges. Case 1 has the KS graph C_{14} (Figure 14) as a subgraph. Figures 35 and 36 show how we found the bipartite complement graph of Case 1 as a subgraph of the bipartite complement graph of C_{14}. In cases 2 and 3, we remove the circled vertices instead, resulting in a $K_{6,6} \setminus 6e$ subgraph, which is IK by Theorem 5.

Note that $K_{7,7} \setminus 10e$ graphs have 39 edges, which is equal to $4V - 17$, providing further support for our conjecture.

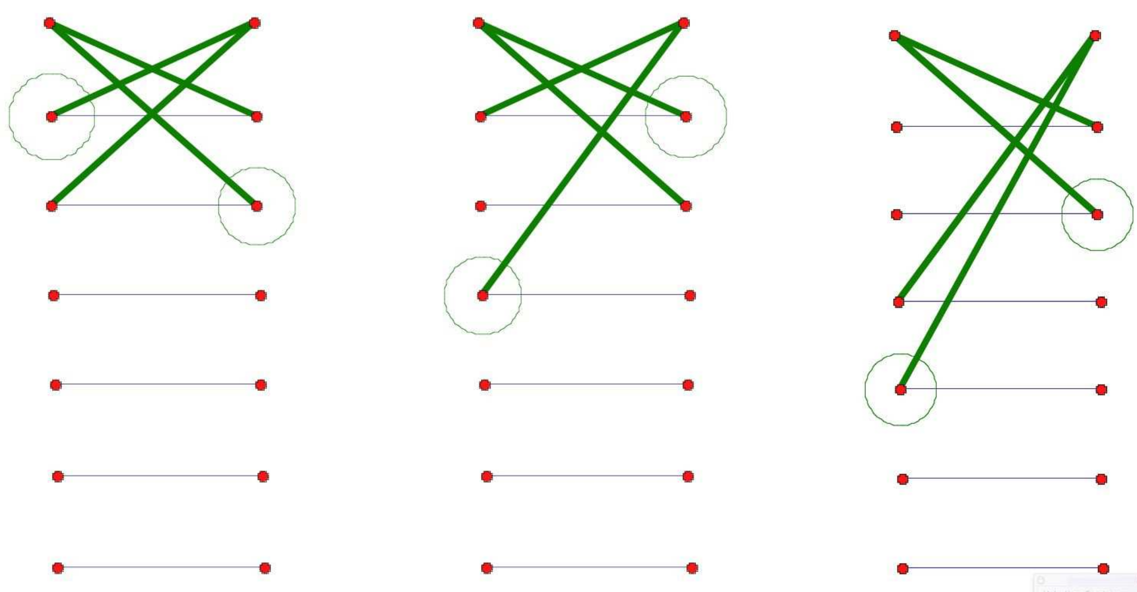


FIGURE 34. There are three ways to remove two vertices from $K_{7,7} \setminus 10e$.

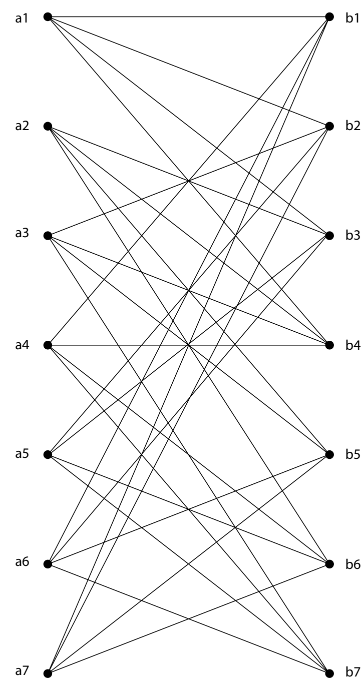


FIGURE 35. The bipartite complement graph of C_{14}

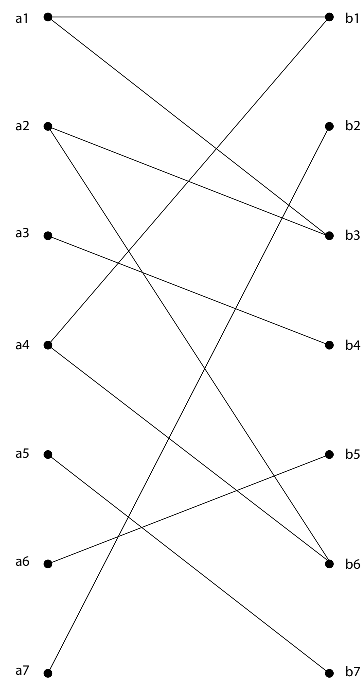


FIGURE 36. The bipartite complement graph of Case 1

CHAPTER III

GENERAL RESULTS

Using the results of the previous chapter, we now provide a sufficient condition that a subgraph of $K_{a,b}$ be intrinsically knotted for each $a, b \geq 6$. This is the content of Propositions 7, 8, and 9, Theorem 10 and Corollaries 11, 17, and 18, which we prove in this chapter.

Proposition 7 (cf. Theorem 5 in [HAM] and Theorem 11 in [HAMM]) will provide more evidence for the solidity of our conjecture. For example, if $n = 1$, then we have a $K_{7,6} \setminus 7e$ graph. Our conjecture would predict that this graph is IK, since $E \geq 4V - 17$. To count for you, a complete $K_{7,6}$ will have $7 \times 6 = 42$ edges, and so a $K_{7,6} \setminus 7e$ graph will have $42 - 7$ or 35 edges. Our inequality, $E \geq 4V - 17$, gives $4 \times 13 - 17$ edges, which is $52 - 17$, or 35. In fact, all of these graphs have exactly 4V - 17 edges.

Proposition 7: Every graph of the form $K_{6+n,6} \setminus (2n + 5)e$ is IK where $n \geq 0$.

Proof:

(by induction on n)

The $n = 0$ case is Theorem 4.

For the base case, let $n = 1$. We will look at three subcases of the base case:

Subcase 1: Consider the $K_{7,6} \setminus 7e$ graph where each of the seven a-vertices has exactly one edge removed, and exactly one of the six b-vertices (call it b_6) has exactly two edges removed compared to the complete graph $K_{7,6}$. We can remove one vertex to show that this graph has a $K_{6,6} \setminus 6e$ subgraph, all of which have been shown to be IK, except G_{666}, in Theorem 5. We can avoid G_{666} by not removing either of the two a-vertices that are not connected to b_6. Removing any of the other a-vertices will leave us with one of the $K_{6,6} \setminus 6e$ graphs which is known to be IK.

Subcase 2: Consider the $K_{7,6} \setminus 7e$ graphs where each of the seven a-vertices has exactly one edge removed and, at least one b-vertex has three or more edges removed, or at least two b-vertices have at least two missing edges each. Remove any a-vertex, and again we are left with a $K_{6,6} \setminus 6e$ minor that is not G_{666}.

Subcase 3: We are left to consider the $K_{7,6} \setminus 7e$ graphs where one or more of the seven a-vertices has more than one edge removed. By removing one of those vertices (with more than one edge removed), we will be left with a $K_{6,6} \setminus me$ minor where $m < 6$, all of which are known to be IK by Theorem 4.

As these three cases cover all possibilities, we have shown that all $K_{7,6} \setminus 7e$ graphs are IK.

Now, for the inductive step, fix $n \geq 1$, and assume all graphs of the form $K_{6+n,6} \setminus (2n+5)e$ are IK. Using Lemma 12 with $a = 6+n$, $b = 6$, $m = 2n+5$, and $k = 2$, we have that every $K_{6+n+1,6} \setminus (2(n+1)+5)e$ graph G is also IK. Thus, by induction, $K_{6+n,6} \setminus (2n+5)e$ is IK for every $n \geq 0$.

Corollary 17 (Corollary 1 in [HAM]; Corollary 12 in [HAMM]):

A bipartite graph with exactly 6 vertices in one part and at least 6 vertices in the other part and $E \geq 4V - 17$ is IK.

Proof:

The complete $K_{6+n,6}$ graph has $36 + 6n$ edges. Therefore a subgraph of the form $K_{6+n,6} \setminus me$, say G, has $E = 36 + 6n - m$ edges. Suppose G has $E \geq 4V - 17$ edges. Since $V = 12 + n$, we have that $E \geq 4n + 31$, whence $m \leq 2n + 5$ and G has a $K_{6+n,6} \setminus (2n+5)e$ subgraph. This subgraph is IK by the preceding proposition and so G is itself IK.

Now we have shown that our conjecture holds at least for the case of the $K_{6+n,6}\backslash me$ graphs. Recall that it was already known for $K_{5+n,5}\backslash me$ graphs [CHPS].

The next proposition (cf. Theorem 6 in [HAM] and Theorem 13 in [HAMM]) gives a similar result for graphs with seven vertices in one part and at least seven vertices in the other part.

Proposition 8: Every graph of the form $K_{7+n,7} \setminus (2n+10)e$ is IK where $n \geq 0$.

Proof:

(by induction on n)

The $n = 0$ case is Theorem 6.

For the base case, let $n = 1$. This gives us a $K_{8,7} \setminus 12e$ graph. We know using Lemma 12 (with $a = b = 7$, $m = 10$, and $k = 2$) and Theorem 6 that every $K_{8,7} \setminus 12e$ graph is IK.

Now, for the inductive step, let $n \geq 1$ and assume all graphs of the form $K_{7+n,7} \setminus (2n+10)e$ are IK. Using Lemma 5 with $a = 7+n$, $b = 7$, $m = 2n+10$ and $k = 2$, we can show that every $K_{7+n+1,7} \setminus (2(n+1)+10)e$ graph is also IK.

Since $K_{7,7} \setminus 10e$ has exactly $4V - 17$ edges, our conjecture is supported in that case. However, this proposition does not allow us to prove our conjecture for all graphs with at least seven vertices in each part. Still, as the following corollary shows, it does improve on the previous best known bound (for graphs in general) [CMOPRW] of $E \geq 5V - 14$.

Corollary 18 (Corollary 2 in [HAM]):

A bipartite graph with exactly 7 vertices in one part and at least 7 vertices in the other part and $E \geq 5V - 31$ is IK.

Proof:

The complete $K_{7+n,7}$ graph has $49 + 7n$ edges. Therefore a subgraph of the form $K_{7+n,7} \setminus me$, call it G, has $E = 49 + 7n - m$ edges. Suppose G has $E \geq 5V - 31$ edges. Since $V = 14+n$, we have that $E \geq 5n+39$, whence $m \leq 2n+10$ and G has a $K_{7+n,7} \setminus (2n+10)e$ subgraph. This subgraph is IK by the preceding theorem and so G is itself IK.

Proposition 9 (Theorem 7 in [HAM]; Theorem 15 in [HAMM]): Every graph of the form $K_{8+n,8} \setminus (2n+15)e$ is IK where $n \geq 1$.

Proof:
(by induction on n)

For the base case, let $n = 1$. This gives us a $K_{9,8} \setminus 17e$ graph. Applying Lemma 12 with $a = 7$, $b = 9$, $m = 14$, and $k = 3$, we can get a $K_{9,7} \setminus 14e$ subgraph, which is IK by Proposition 8, so our graph is IK.

Now, for the inductive step, let $n \geq 1$ and assume all graphs of the form $K_{8+n,8} \setminus (2n+15)e$ are IK. Using Lemma 12 with $a = 8+n$, $b = 8$, $m = 2n+15$, and $k = 2$, we can show that every $K_{8+n+1,8}$ $(2(n+1)+15)e$ graph is also IK.

We don't know whether or not Proposition 9 holds when $n = 0$. We do, however, know that all $K_{8,8} \setminus 14e$ graphs are IK (use that all $K_{8,7} \setminus 12e$ graphs are IK, by Proposition 8, and Lemma 12 with $k = 2$, $m = 12$, $a = 7$, and $b = 8$,).

So far, we've given conditions for IK of $K_{a,a+n}$ graphs with $a = 5, 6, 7$ or 8. We conclude with a theorem and corollary that characterize intrinsic knottiness for graphs where $a \geq 9$.

Theorem 10 (Theorem 8 in [HAM]): Every graph of the form $K_{a,a} \setminus (6a-34)e$ is IK where $a \geq 9$.

Proof:

We use induction on a. Consider a $K_{9,9} \setminus 20e$ graph. We have shown, through Proposition 9, that all $K_{9,8} \setminus 17e$ graphs are IK. By Lemma 12, with $k = 3$, $m = 17$, $a = 8 = b - 1$, if all $K_{9,8} \setminus 17e$ graphs are IK, then every $K_{9,9} \setminus 20e$ graph is also IK.

Now, for the inductive step, let $a \geq 9$ (we'll need this in applying Lemma 12, below) and assume all graphs of the form $K_{a,a} \setminus (6a-34)e$ are IK. We will show that every $K_{a+1,a+1} \setminus (6(a+1)-34)e$ graph G is IK.

We have assumed that all graphs of the form $K_{a,a} \setminus (6a-34)e$ are IK, where a is the number of vertices in each part. Set $k=3$, $m=6a-34$, and $a=b$. Since $(k-1)(a+1) = 2(a+1) < 6a-31 = m+k$, we have by Lemma 12 that every $K_{a+1,a} \setminus (6a-31)e$ graph is IK. Applying the lemma one more time with $k=3$, $m=6a-31$, and $b=a+1$ gives us that all $K_{a+1,a+1} \setminus (6a-28)e$ graphs are IK. Since $6a-28$ is $6(a+1)-34$, this means that every $K_{a+1,a+1} \setminus (6(a+1)-34)e$ graph is IK.

Thus, by induction, every graph of the form $K_{a,a} \setminus (6a+34)e$ is IK for all $a \geq 9$.

Corollary 11 (Theorem 9 in [HAM]; Theorem 17 in [HAMM]): Every graph of the form $K_{a+n,a} \setminus (3n+6a-34)e$ is IK where $a \geq 9$ and $n \geq 0$.

Proof:

(by Induction on n)

For the base case, let $n=0$. This gives us a $K_{a,a} \setminus (6a-34)e$ graph. This is IK by Theorem 10.

For the inductive step, assume that all $K_{a+n,a} \setminus (3n+6a-34)e$ graphs are IK. Now, look at a $K_{a+n+1,a} \setminus (3(n+1)+6a-34)e$ graph. This is also a $K_{a+n+1,a} \setminus (3n+6a-31)e$ graph. By Lemma 12, $K_{a+n+1,a} \setminus (3n+6a-31)e$ graphs are IK provided $(3n+6a-31)$ is more than twice $(a+n+1)$.This is true when $a \geq 9$, by induction on a. When $a=9$, $3n+23$ is larger than $n+10$. Each time a increases by one, $3n+6a-31$ will increase by 6 while $2(a+n+1)$ will only increase by 2. So, by induction, the inequality for Lemma 12 holds and every $K_{a+n+1,a} \setminus (3(n+1)+6a-34)e$ graph is IK.

CHAPTER IV

CONCLUSIONS AND FURTHER QUESTIONS

Now we summarize the results of our research and look at remaining questions and topics for further research.

We began by discussing the general knowledge needed to understand the elements of knot theory, graph theory (especially bipartite graphs) and intrinsic knottiness that were used in our research.

This paper was motivated by the possibility of defining a bound that would predict intrinsic knottiness for a general class of bipartite graphs

Motivating Conjecture:
Every bipartite graph that has at least five vertices in each part, and that meets the condition $E \geq 4V - 17$, is intrinsically knotted.

We gathered support for our motivating conjecture by proving some specific results. We began by showing that our bound cannot be improved to $E = 4V - 18$, because three of the four $K_{5,5} \setminus 3e$ (for which $E = 4V - 18$) graphs are not IK. We then proved that any graph of the form $K_{6,6} \setminus 5e$ (for which $E = 4V - 17$) is IK. Similary, we proved that any graph of the form $K_{6,6} \setminus 6e$ is IK, with the possible exception of G_{666}. We also showed that graphs of the form $K_{7,7} \setminus 10e$ are IK.

We then moved on to generalizing our results. We showed that all graphs of the form $K_{6+n,6} \setminus (2n+5)e$ are IK where $n \geq 0$. This result proved our conjecture that if $E \geq 4V - 17$, then the graph is intrinsically knotted for the case of a graph with exactly six vertices in one part and at least six in the other. In addition, our conjecture was verified for graphs with exactly seven vertices in each part.

We were not able to prove or disprove our conjecture with full generality. We were able to give bounds on the number of edges required to ensure intrinsic knottiness for any bipartite graph, but the question of whether a bipartite graph with at least five vertices in one part and $E \geq 4V - 17$ is intrinsically knotted is still open.

Our research was inspired by the result that all graphs with $E \geq 5V - 14$ and at least seven vertices are IK [CMOPRW]. That research showed that by starting with a planar graph and adding two vertices, they could produce many bipartite graphs with $5V - 15$ edges that are not IK. This shows that their bound is the best possible.

Question 1:

Is $E \geq 4V - 17$ the correct bound? If not, how can we find non-IK bipartite graphs with $4V - 17$ edges?

If $E \geq 4V - 17$ is the correct bound, then we would expect to find an infinite family of non IK graphs for which $E = 4V - 18$. But we could find only three examples—the three graphs of the form $K_{5,5} \setminus 3e$ that are not IK. In Chapter 2, we applied Proposition 16 to obtain intrinsically knotted bipartite graphs for which $E > 4V - 17$. However, the contrapositive of this process cannot be used to make non IK graphs with $E = 4V - 17$. If we cannot do this, it might be better to think of these three $K_{5,5} \setminus 3e$ non IK graphs as exceptions to a bound $E \geq 4V - 18$. For example, using Naimi's calculation [N], we find all $K_{6,6} \setminus 6e$ (for which $E = 4V - 18$) are IK.

Question 2:

Can we calculate a similar bound for other families of graphs?

We extended the research of [CMOPRW] to bipartite graphs, and were able to improve upon their bound of $E > 5V - 14$ by restricting to a smaller class of graphs. Are there other families where we could calculate such a bound?

Would research into multipartite graphs or other families of graphs provide an even better bound for those families of graphs?

Question 3:

Is G_{666} intrinsically knotted?

Another looming question that remains is the IK status of G_{666}. If this particular graph could be shown to be IK (computer experiments by Naimi [N] suggest that it is), then we could prove all $K_{6,6} \setminus 6e$ graphs are IK. This would mean $E \geq 4V - 18$ is a sufficient condition for bipartite graphs with six vertices in each part.

Finally, we remark that while we failed to show $E \geq 4V - 17$ ensures IK for bipartite graphs, in [HAMM] we did prove a bound of the form $E \geq 4V - C$, where C is a constant. Specifically, we proved the following theorem:

Theorem 19:

Let a_n be defined by the recurrence

$$a_n = \left\lfloor \frac{n(a_{n-1} - 1)}{n - 5} \right\rfloor + 1$$

when $n \geq 7$, $a_5 = 5$, and $a_6 = 7$. Let $C_n = a_n - 4n$, for $n \geq 7$, and $C_5 = C_6 = -17$. Let G be a bipartite graph with exactly $n \geq 5$ vertices in one part and at least a_n vertices in the other. If $e(G) \geq 4v(G) + C_n$ then G is IK.

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