MAXIMALLY KNOTLESS GRAPHS

A Thesis

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to the Faculty of

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In Partial Fulfillment

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Master of Science

in

Mathematics Education

---

by

Lindsay Eakins

Summer 2020

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DEDICATION

I would like to dedicate this thesis to my mother, Kathi, my brother, Nolan, and my loving husband, Loren, who have always believed in me, even when I did not believe in myself. Without your love and encouragement I would not be where I am today.

ACKNOWLEDGEMENTS

First and foremost, I would like to acknowledge my Thesis Chair, Thomas Mattman, who has been extrememly supportive and encouraging over these past few months while working on this thesis. I could not have accomplished this without your guidance and timely feedback. I truly appreciate your dedication and efforts in helping me complete my work.

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ABSTRACT

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This thesis explores properties of maximally knotless graphs. We prove: (1) if G is maximally knotless with $|V| \geq 3$ vertices, then G is connected and has minimum degree 2; (2) for $|V| \geq 7$ vertices, maximally knotless graphs have between 20 and $5n - 15$ edges; and (3) a 2-apex graph is maximally knotless if and only if it is maximally 2-apex. We proceed to classify maximally knotless graphs through nine vertices and twenty edges. Lastly, using the graph E_9, a graph with 9 vertices and 21 edges, we prove: (1) the existence of a maximally knotless graph with 15 vertices and 39 edges; (2) the existence of a maximally knotless graph with n vertices and $m < 3n$ edges; and (3) if G is a maximally knotless graph with $n \geq 4$ vertices and m edges, then $m \geq \frac{3n}{2}$.

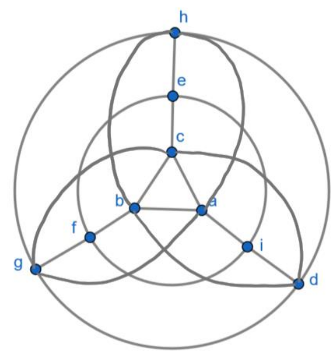
CHAPTER I

INTRODUCTION

This chapter introduces the foundational information needed to understand the statements of the Theorems that will be proven. This includes a definition of the graph E_9 and its properties and basic terminology from graph theory and knot theory. After stating the theorems, the chapter concludes with an overview of the structure of the thesis including a summary of each chapter. This is a first study of maximally knotless graphs, inspired by the work of Max Aires (2019) on maximally linkless graphs. Similarly to that paper, we develop a lower bound for the number of vertices and edges in maximally knotless graphs. In addition, we classify the maximally knotless graphs through nine vertices and twenty edges. In this first chapter, some terminology will be introduced without complete definitions (e.g. "graph", "vertices", and "edges") in order to get to a statement of the main results. More thorough definitions will follow in Chapter 2.

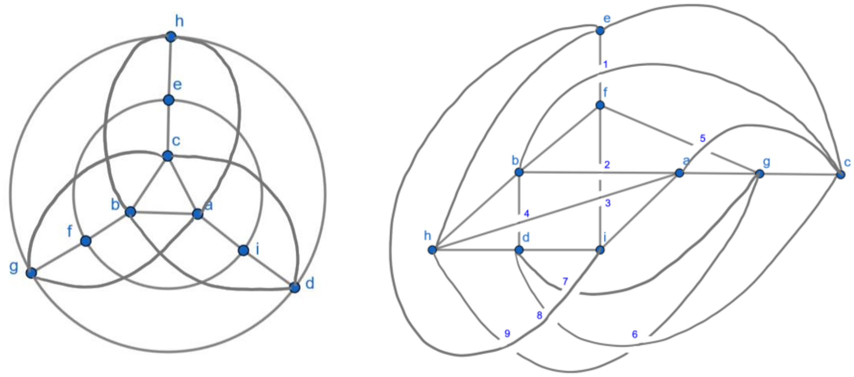
The Graph E_9

A **graph** is a set of **vertices,** V, and a set of **edges,** E. An important example for this thesis is E_9, a graph of 9 vertices and 21 edges, illustrated below. Here, V = {a,b,c,…,i} and edges are represented by curves connecting pairs of vertices. This image shows that E_9 has a lot of symmetry.



Graph Embeddings and Knots

An **embedding of a graph**, G, is a representation of the graph in R^3. Vertices are points and edges are curves between those points. Below are two illustrations of E_9. The one on the left shows edges running through each other and is not an embedding. In contrast, the illustration on the right shows an embedding. Note that there are breaks in the lower curve where edges cross to show it is below. The crossings are labelled 1 to 9.



A **cycle** is a simple path in the graph that begins and ends at the same vertex. This means a cycle is an **embedding of a circle** in R^3, in other words, a **knot**. The unknot (left) and the Trefoil knot (right) are two examples of knots.

Maximally Knotless Graphs

An embedding of a graph is called **knotless** if all cycles are unknots. In Chapter 3, we will show that the embedding of E_9 given above is knotless. A graph is **intrinsically knotted (IK)** if, no matter how it is embedded in R^3, it contains a knotted cycle.

A graph, G, is **maximally knotless**, if G has a knotless embedding and whenever an edge uv, not already in G, is added, the new graph is IK. This is a new definition in the literature and the focus of this thesis is to study maximally knotless graphs.

Main Theorems: Theorem 1 – 11

The following Theorems will be proved in Chapter 3:

I. **Preliminary Theorems**

Theorem 1: Let G be a maximally knotless graph. Then G is connected. If $|V| \geq 3$, then G has minimum degree ≥ 2.

Theorem 2: For $|V| = n \geq 7$ vertices, a maximally knotless graph has between 20 and $5n - 15$ edges.

Theorem 3: A 2-apex graph is maximally knotless if and only if it is maximally 2-apex.

II. **Classification of Maximally Knotless Graphs through Nine Vertices and Twenty Edges**

Theorem 4: For $n < 7$, the complete graph K_n is the only maximally knotless graph on n vertices. The complete graph with an edge removed, K_7^-, is the only maximally knotless graph on seven vertices. These seven graphs are the maximally knotless graphs with at most twenty edges.

Theorem 5: There are two maximally knotless graphs on 8 vertices.

Theorem 6: There are seven maximally knotless graphs on 9 vertices.

Theorem 7: The graph $G_{(9,29)}$ is maximally knotless.

Theorem 8: The graph E_9 is maximally knotless.

III. **Number of Edges in a Maximally Knotless Graph**

Theorem 9: There is a maximally knotless graph with 15 vertices and 39 edges.

Theorem 10: There exist maximally knotless graphs with n vertices and $m < 3n$ edges for arbitrarily large n.

Theorem 11: Let G be a maximally knotless graph with $n \geq 4$ vertices and m edges. Then, $m \geq \frac{3n}{2}$.

Classification of Maximally Knotless Graphs Through Nine Vertices and Twenty Edges

Together, Theorems 4 to 8 classify the maximally knotless graphs on nine or fewer vertices. There are exactly 16 graphs. All of them are maximally 2-apex (see Chapter 2, Apex Graphs for a definition) except for E_9 and the graph $G_{(9,29)}$ (see Theorem 7 in Chapter 3). These examples allow us to test the generality of the more general statements that follow them. They are the inspiration for several of our suggestions for further research in Chapter 4. In addition, we show there are exactly seven maximally knotless graphs with twenty or fewer edges.

Number of Edges in Maximally Knotless Graphs

In defining maximally knotless, we were inspired by Aires (2019) who investigated a similar notion of maximally linkless. There are others who have studied maximally linkless graphs, including Sachs (1984), Jørgensen (1989), Maharry (1998), Dehkordi and Farr (2019) and Naimi, Pavelescu, and Pavelescu (2020). These ideas are related to maximally planar, a classical topic in graph theory. This is because an intrinsically knotted (IK) graph is intrinsically linked (IL) and an IL graph is non-planar. However, just as researchers observed for maximally linkless, the behavior of the number of edges in maximally knotless graphs is very different compared to graphs that are maximally planar.

Like maximally knotless, a maximally planar graph contains the maximum number of edges while remaining planar. In Chapter 2 (Planarity), we will observe that maximally planar graphs have exactly $3|V| - 6$ edges (assuming $|V| \geq 3$).

As discussed in Chapter 3, Theorems 2 and 11, in a maximally knotless graph, the number of edges ranges from $\frac{3|V|}{2}$ to $5|V| - 15$ which is notably different from the behavior for maximally planar graphs. Aires (2019) and other researchers made a similar observation for maximally linkless graphs. Note that Theorem 4 gives a family of graphs that have $3|V| - 6$ edges, which is the same as maximally planar! Since IK implies IL implies non-planar, we might expect that knotting and linking ought to mean more edges than non-planar graphs, but that is not always true.

To summarize, this thesis will provide information on the number of edges in maximally knotless graphs and classify the maximally knotless graphs through nine vertices and twenty edges. Chapter 2 will explain basic definitions of graph theory and knot theory, as well as other important theorems and concepts, providing the basis for the eleven theorems outlined above. Chapter 3 will include proofs for each of the preliminary threorems, theorems on the classification of graphs through nine vertices and twenty edges, and threorems on the number of edges in a maximally knotless graph. Chapter 4 will conclude the thesis with a review of our findings and some suggested areas for further research in graph theory and knot theory. Appendix A provides a detailed exploration of cycles in a knotless embedding of the graph E_9 that include two consecutive, non-alternating crossings. Appendix B provides further exploration of the remaining cycles in the knotless embedding of the graph E_9 that have at least three crossings, after removing the combinations of crossings included in Appendix A.

CHAPTER II

LITERATURE REVIEW

In this chapter we review known material that will be used to build a foundation for the proofs of our theorems in Chapter 3. We will begin with basic definitions used in graph theory and knot theory. Additionally, we will discuss two families of graphs, Petersen and Heawood, useful for understanding knotting and linking of graphs.

We begin with an overview of some basic ideas in graph theory. A good reference for the material in the next few sections is Marcus (2008).

Graphs and Degree of a Vertex

A **graph**, G, is composed of a finite set of **vertices,** V, and a finite set of **edges,** E. Each edge has two vertices as endpoints and will be denoted ab, with $a, b \in V$. We will assume graphs are undirected, meaning the edges ab and ba are equivalent. The **degree** of a vertex is equal to the number of edges that include that vertex.

Minimum Degree of a Graph

The **minimum degree** of a graph, G, is equal to degree of the vertex with the least number of edges incident to it. All other vertices in the graph must have degree equal or greater than that of a vertex with minimum degree.

Complete Graphs

A **complete graph**, G, has all vertices adjacent to each other. A complete graph is denoted by K_n where n is the number of vertices. The degree of each vertex in a complete graph is the same. See Figure 2.2 for illustrations of complete graphs K_1 through K_4.

Complement of a Graph

The **complement** of a graph, G, is a graph, H, such that G and H share the same set of vertices but the edges in H are the edges of the complete graph that are not included in G. In *Figure 2.1* below, the graph G has 5 vertices and 5 edges. Adding in the red edges to form the complete graph, K_5, gives us the complement, H, with 5 vertices and 5 edges.

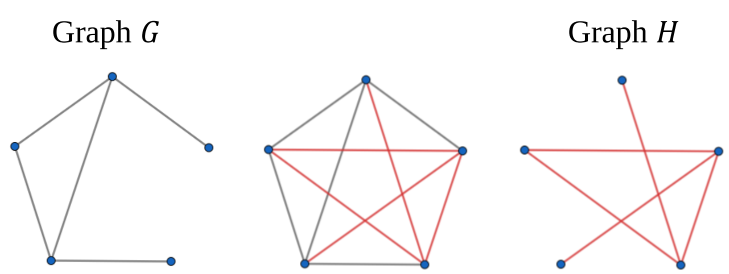


Figure 2.1: Complement of a Graph

Handshake Lemma

The **Handshake Lemma** states that the total degree sum of the vertices in a graph is twice the number of edges: $\sum_{v \in V} \deg v = 2|E|$. This is sometimes called Euler's theorem since it is mentioned in the very first paper in graph theory about the Königsberg bridges.

Planarity

A graph, G, is said to be **planar** if it can be drawn in R^2 without any crossings. Examples of planar graphs include the complete graphs K_1, K_2, K_3, and K_4 since these graphs can be drawn in R^2 without any crossing of edges.

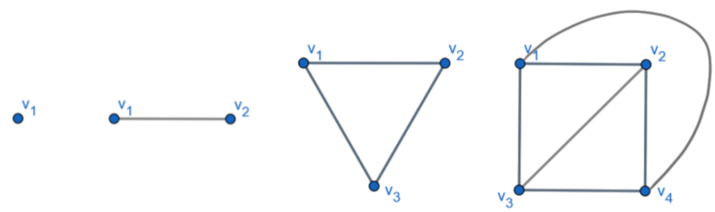


Figure 2.2: Complete Graphs K_1 Through K_4

It turns out that the complete graph K_5 is **non-planar**. No matter how it is drawn in R^2 there is at least one crossing.

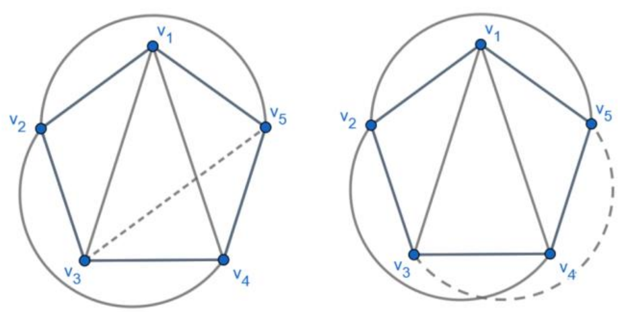


Figure 2.3: Exploring Crossings in K_5

Another non-planar graph is the bipartite graph $K_{3,3}$ which consists of two sets of three vertices such that each vertex is adjacent to all vertices in the other set. No matter how $K_{3,3}$ is drawn in R^2 there exists at least one crossing.

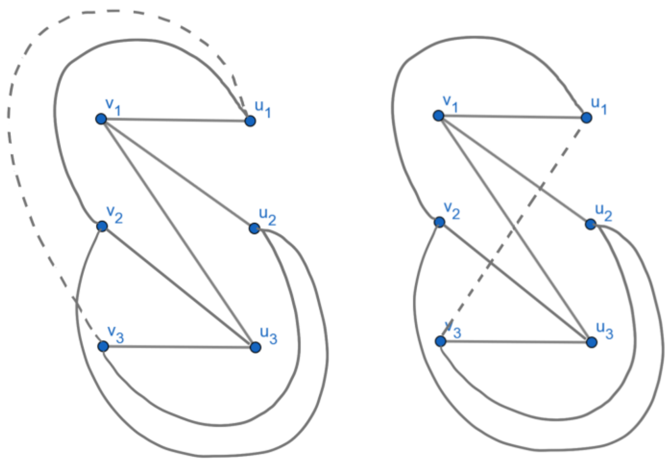


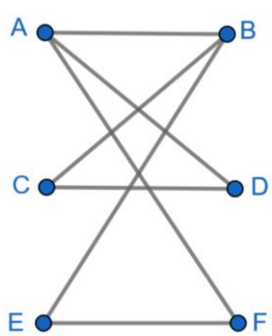
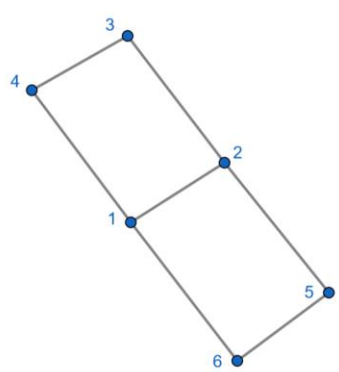
Figure 2.4: Exploring Crossing in $K_{3,3}$

Kuratowski's Theorem (Kuratowski, 1930) states a graph is planar if and only if does not contain a K_5 or $K_{3,3}$.

Isomorphic Graphs

Two undirected graphs, G and H, are **isomorphic** if there exists a bijection, f, between the sets of vertices in G and H so that ab is an edge of G if and only if $f(a)f(b)$ is an edge of H. Although the illustrations of the graphs may look entirely different, the graphs in *Table 2.1* are isomorphic.

Table 2.1: Isomorphic Graphs

Graph *G*	**Graph *H***	**Isomorphism Between *G* and *H***
		$f(A) = 1$ $f(B) = 2$ $f(C) = 3$ $f(D) = 4$ $f(E) = 5$ $f(F) = 6$

Apex Graphs

An **apex graph** (left) is either planar, or becomes planar after one of the vertices (and all its edges) is deleted. A graph is **2-apex** if it is planar, apex, or becomes planar after two vertices are deleted. This means an apex graph can be embedded in R^3 (See Chapter 1 for a definition of graph embedding) so that one vertex is suspended above the rest of the graph which lies in a plane. In a 2-apex graph (right), there is an embedding such that one vertex is above the plane containing the remainder of the graph and one vertex is below the plane.

2-Apex Graph

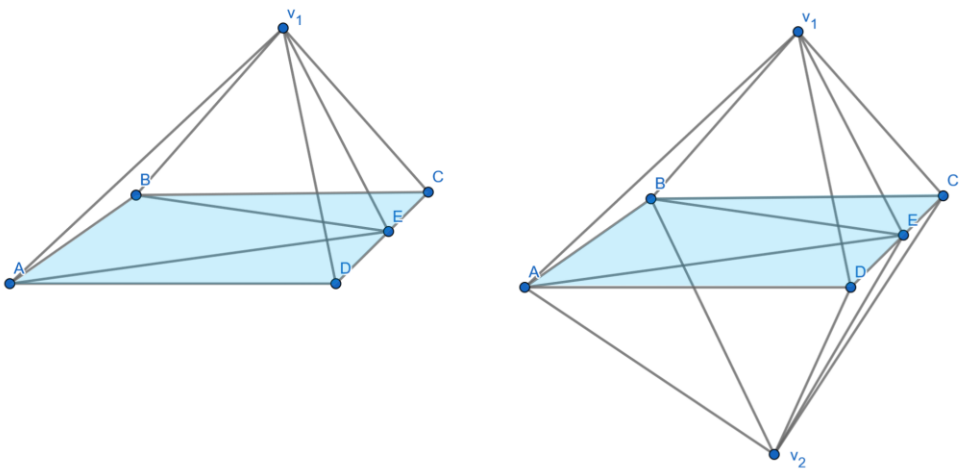


Figure 2.5: Apex and 2-Apex Graphs

It is important to note the differences in the number of edges in maximally planar, maximally apex and maximally 2-apex graphs. As mentioned in Chapter 1, for $|V| \geq 3$, maximally planar graphs have $3|V| - 6$ edges. For $|V| \geq 4$, maximally apex graphs (graphs that are apex until an edge, not already in G, is added to make a non-apex graph) have $4|V| - 10$ edges. For $|V| \geq 5$, **maximally 2-apex** graphs (graphs that are 2-apex until an edge, not already in G, is added to make a non-2-apex graph) have $5|V| - 15$ edges.

Paths and Cycles

A **path** in a graph is a sequence of vertices and edges between them such that vertices are not repeated. In *Figure 2.6* below, an example of a path is: $v_1, v_1v_2, v_2, v_2v_5, v_5, v_5v_3, v_3, v_3v_4, v_4$. A **cycle** is a closed path in a graph, G; thus, it begins and ends at the same vertex and otherwise no vertices are repeated. In *Figure 2.6* below, an example of a cycle is: v_1, v_1v_2, v_2, v_2v_3, v_3, v_3v_4, v_4, v_4v_5, v_5, v_5v_1, v_1. For short, we denote this cycle $\{v_1, v_2, v_3, v_4, v_5, v_1\}$.

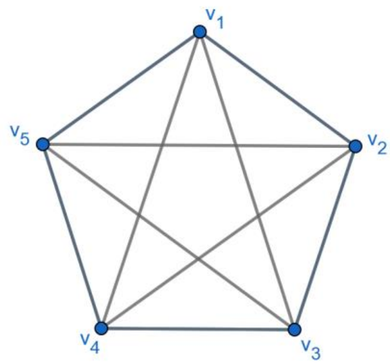


Figure 2.6: Path and Cycle in a Graph

Connected Graphs and Components

A graph, G, is **connected** when there exists a path between every pair of vertices. A connected graph has one **component** (left). When a graph is not connected, it has more than one component (right). This means that there are at least one pair of vertices that have no path between them.

Connected Graph
with One Component

Not Connected Graph
with Two Components

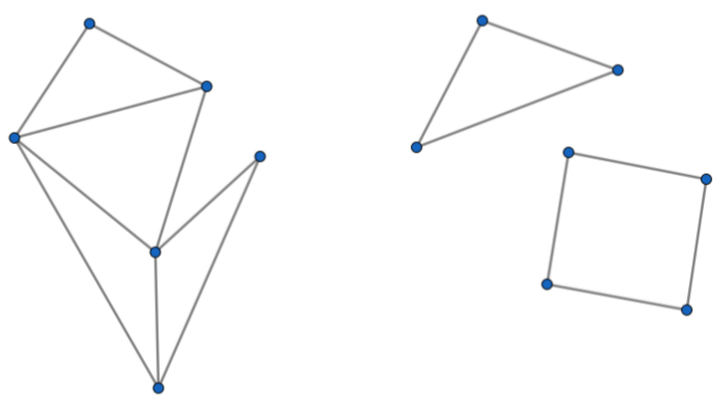


Figure 2.7: Connected and Not Connected Graphs

A good reference for the material in the next few sections about linking and knotting of graphs is Adams (2004).

Cycles and Linking

In an embedding of a graph G, if two cycles, C_1 and C_2, exists in G such that the edges of C_1 and C_2 are disjoint and the cycles lie in the same plane, the cycles are **unlinked**. More generally, if the embedding can be deformed continuously through space, so that edges never pass through one another, to make the cycles disjoint in a plane, then the cycles are unlinked. If this is not possible, we say the cycles are (nontrivially) **linked**. In *Figure 2.8* below, the embedding on the left depicts unlinked cycles. The embedding on the right depicts (nontrivially) linked cycles.

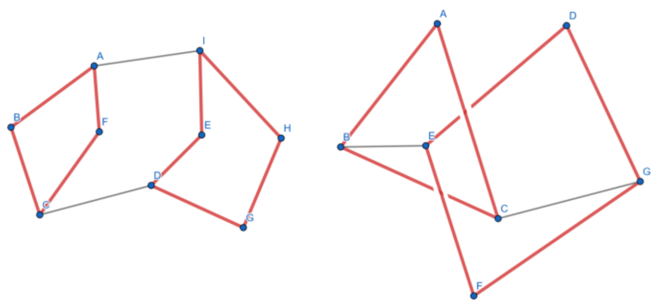


Figure 2.8: Cycles and Linking

Linkless and Intrinsically Linked Graphs

An embedding in R^3 of an undirected graph, G such that no two cycles of the graph form a nontrivial link is called a **linkless embedding**. A graph that does not have a linkless embedding is called an **intrinsically linked (IL)** graph. For example, planar and apex graphs, have linkless embeddings. The complete graph K_6 is an example of an IL graph.

$\mathrm{Y} - \Delta$ Transformations

A $\boldsymbol{Y - \Delta}$ **transformation** is the replacement of a Y subgraph in a graph, G, with a Δ subgraph. To do this, a vertex, u, of degree 3 is removed from G. Three edges are then added to G to connect the three vertices $v_1, v_2,$ and v_3 that were adjacent to u. See *Figure 2.9*. This transformation preserves the number of edges in G but reduces the number of vertices by one. The opposite move (from right to left in the figure) is called a $\boldsymbol{\Delta - Y}$ **transformation**. Two graphs are considered $\boldsymbol{Y - \Delta}$ **equivalent** if one can be obtained by the other through a series of $Y - \Delta$ or $\Delta - Y$ transformations. We call the set of graphs $Y - \Delta$ equivalent to G, the $\boldsymbol{G}$ **family**. (Goldberg et al,, 2014)

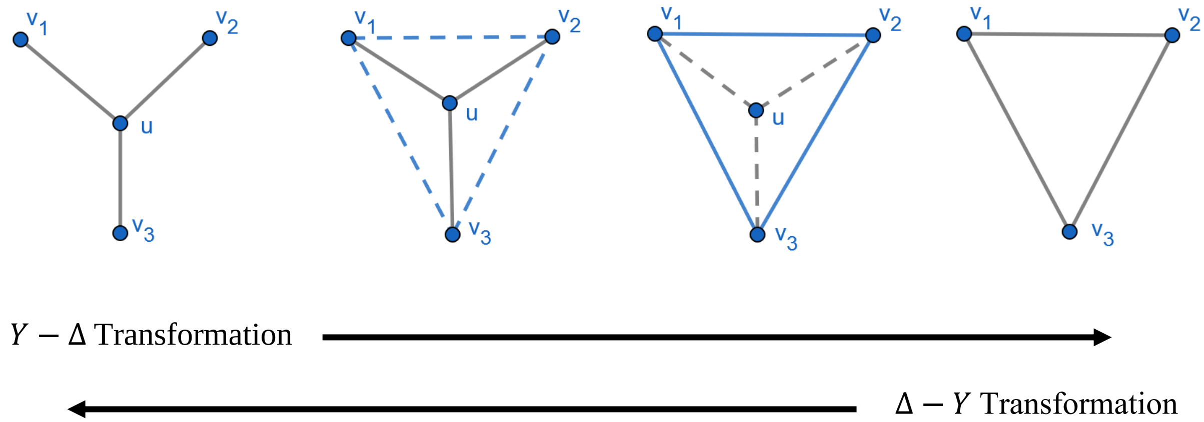


Figure 2.9: $Y - \Delta$ Transformations

The Petersen Family Graphs

The **Petersen Family graphs** consist of the complete graph K_6, the Petersen graph, and five other graphs as seen in *Figure 2.10*. The complete graph K_6 is located at the top left of the figure and the Petersen graph is located at the bottom right of the figure. Each blue line is either a $\Delta - Y$ (add a vertex) or $Y - \Delta$ (remove a vertex) transformation. Each of these graphs is intrinsically linked. That is, they admit no linkless embedding. The Petersen Family graphs are the graphs that are $Y - \Delta$ equivalent to the Petersen graph. Just as the planar graphs are those that have no K_5 or $K_{3,3}$, the Petersen Family graphs characterize linkless embeddable graphs: A graph is IL if and only if it includes a graph from the Petersen family (Robertson, Seymour and Thomas, 1993).

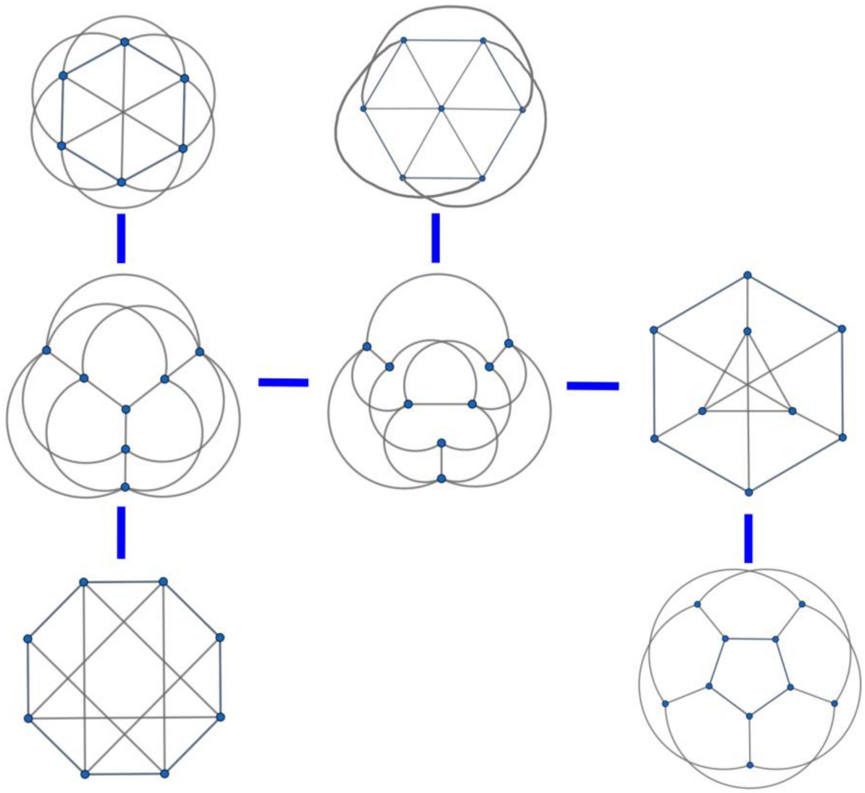


Figure 2.10: The Petersen Family Graphs

Knots, Reidemeister Moves, and Equivalent Knots

A good reference for this section is Adams (2004).

A **knot** is an embedding of a circle S^1 into R^3: $f: S^1 \rightarrow R^3$. By abuse of notation, we often also call $f(S^1)$ the knot. A **knot diagram** is a projection of a knot into R^2 that includes information about the crossing of edges in R^3. A **crossing** in a knot diagram is where the curve passes over or under itself and is depicted by a break in the understrand. For example, crossing 1 from *Figure 2.11* depicts the vertical curve passing over the horizontal curve.

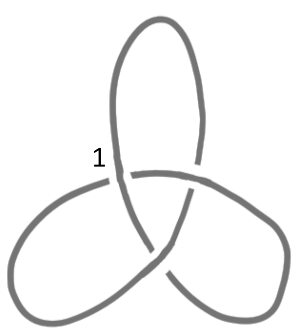


Figure 2.11: Knot Diagram Crossings

Two knots are considered to be **equivalent** if we can smoothly deform one to the other without ever having to pass the curve through itself. To check for equivalent knots we can use three types of transformations of the knot diagram known as **Reidemeister moves**. A **Type I move**, a twist, is when a strand is twisted or untwisted in either direction.

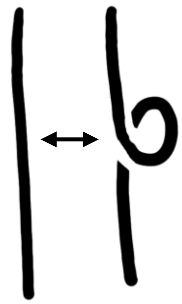


Figure 2.12: Reidemeister Move Type I

A **Type II move**, a poke, is when one strand is moved completely over (or under) another.

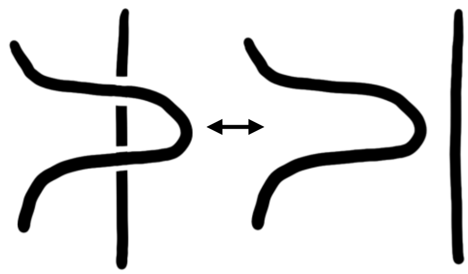


Figure 2.13: Reidemeister Move Type II

A **Type III move**, a slide, is when a strand is moved completely over or under a crossing.

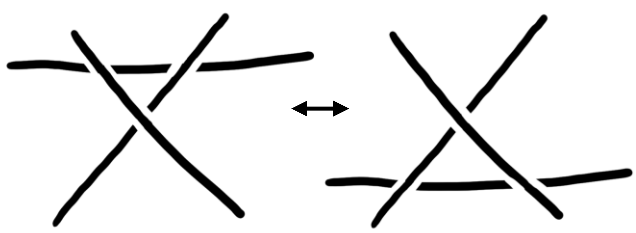


Figure 2.14: Reidemeister Move Type III

Theorem: Two knot diagrams represent equivalent knots if and only if one can be transformed to the other by a sequence of Reidemeister moves.

A **trivial knot**, or an unknot, is one that's equivalent to a unit circle. The simplest non-trivial knot, also known as a **trefoil knot** or 3_1, has a knot diagram with three **alternating crossings**. This means over and under crossings alternate as we trace along the knot. Note that if a knot has a projection with fewer than three crossings, Reidemeister moves can be used to show it is equivalent to the unknot. Similarly, a projection with three crossings that are not alternating is the unknot. A knot with a projection with four alternating crossings (that has no three crossing diagram) is known as a **figure-8 knot** or 4_1. Many more knots exist with five or more crossings.

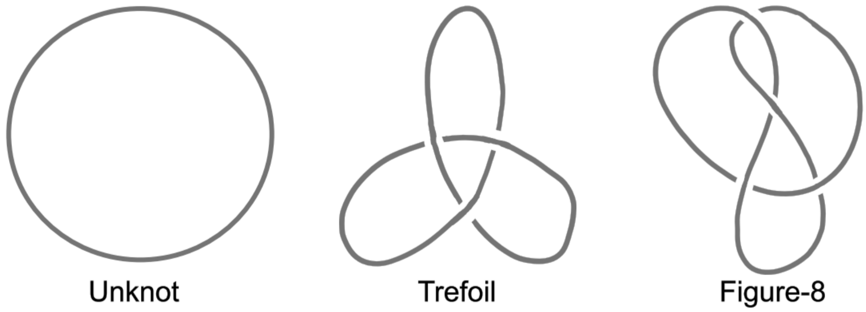


Figure 2.15: Examples of Knots

Knotless, Intrinsically Knotted, and Maximally Knotless Graphs

An embedding in R^3 of a graph, G, that does not contain any cycles that are non-trivial knots is called a **knotless embedding**. A graph that does not have a knotless embedding is called an **intrinsically knotted (IK)** graph. A graph is **maximally knotless** if it has a knotless embedding and is edge-maximal for this property. That is, adding any edge that is not already in the graph results in an IK graph. Unlike the other material in this chapter, the idea of maximally knotless is a new notion that we are introducing in this thesis. It is inspired by the study of maximally linkless graphs. It is also important to note that the embedding of a 2-apex graph described in Chapter 2, Apex Graphs is knotless (Blain et al. 2007 and Ozawa and Tsutsumi 2007).

Edge Contraction

Edge contraction is a way of transforming a graph to one with one fewer vertices. Let edge e have endpoints v_1 and v_2. We delete edge e and replace the two vertices with a single vertex v. Each vertex adjacent to v_1 or v_2 becomes adjacent to v.

The following figure (based on one in Huck, 2010) gives an example.

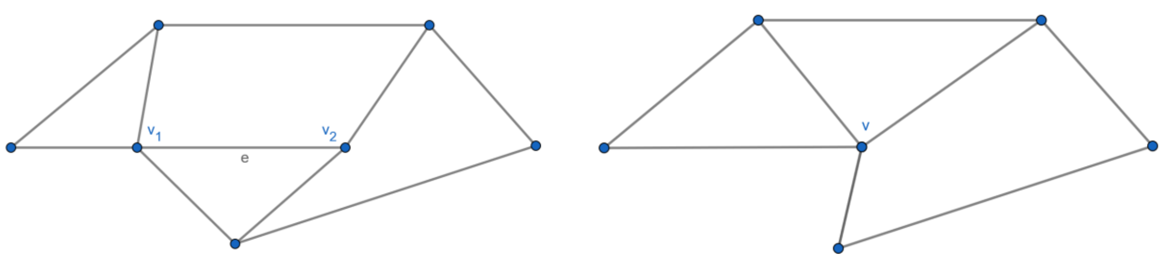


Figure 2.16: Edge Contraction

Minor of a Graph

A **minor**, H, can be formed from a graph, G, by deleting vertices and edges or by contracting edges. A graph G inherits many of the properties discussed above from any minor H: non-planar, IL, IK, not apex or not 2-apex. For example, if H is intrinsically knotted (IK), then G is also IK. Conversely, if G has any of the following properties, the same is true of every minor H: planar, has a linkless embedding, has a knotless embedding, apex or 2-apex. For example, if G has a knotless embedding, then H has one as well (See Goldberg et al., 2014, for example).

The Heawood Family Graphs

Similar to the Petersen Family, the **Heawood Family** (shown in *Figure 2.16*) contains the complete graph K_7, the Heawood graph, and 18 other graphs that are obtained by $Y - \Delta$ or $\Delta - Y$ transformations. The complete graph K_7 is located at the top of the figure and the Heawood graph is located at the bottom of the figure (and labelled C_{14}). The set of six N_i graphs are known to have knotless embeddings (Goldberg et al., 2014; Hanaki et al., 2011). All other graphs in this family are intrinsically knotted (Kohara & Suzuki, 1992). It is important to note that the graph N_9 is an isomorphism of the graph E_9 mentioned in Chapter 1, a graph of central importance in this thesis. The graph F_9 will be useful in proving Theorem 8 in Chapter 3.

The complete graph K_7 is the simplest IK graph. This means that no proper subgraph of K_7 is IK (Conway and Gordon, 1983).

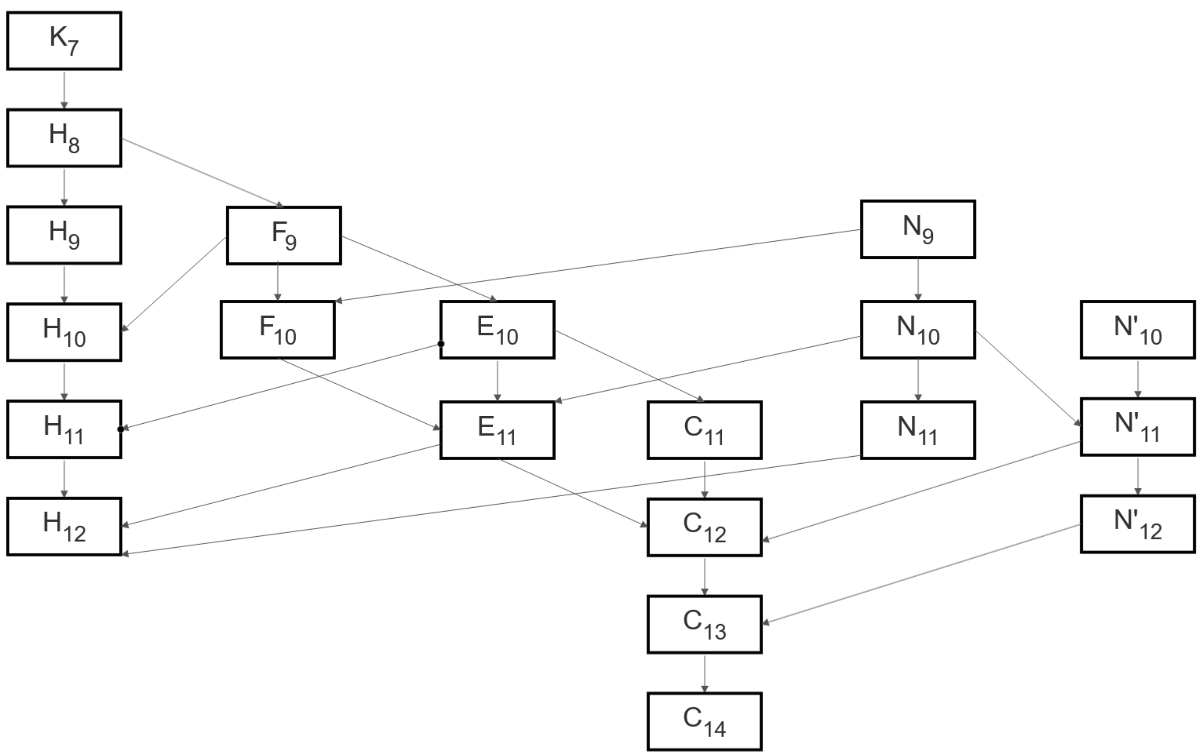


Figure 2.17: The Heawood Family Graphs

The Knot Sum Theorem

Given diagrams of two knots K_1 and K_2, we define a new knot, called the **knot sum**, by removing a small arc from each knot projection and then connecting the four endpoints by two new arcs.

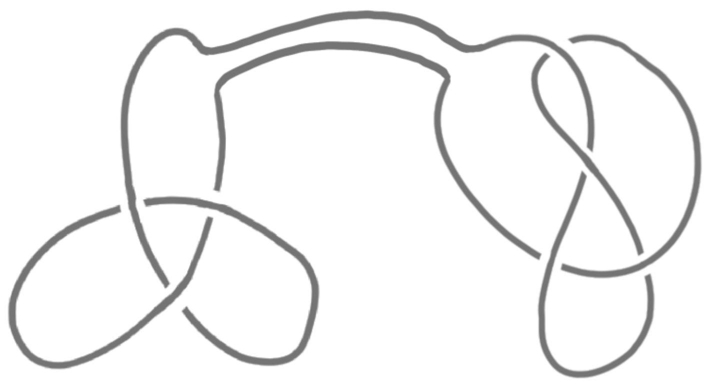


Figure 2.18: Knot Sum

Knot Sum Theorem: The knot sum of any knot K with the unknot results in the same knot K. Furthermore, the knot sum of any number of knots cannot be the unknot unless each knot summand is the unknot.

CHAPTER III

PROOFS

In this chapter we will prove the eleven theorems stated in Chapter 1. We begin with some general observations, Theorems 1 to 3, regarding maximally knotless and maximally 2-apex graphs. Theorems 4 through 8 give a classification of maximally knotless graph for $|V| \leq 9$ vertices and $|E| \leq 20$ edges. Lastly, Theorems 9 through 11 closely follow the work of Aires (2019). In Theorem 8, we prove that the graph E_9 is maximally knotless and in Theorems 9 and 10 we use that to generate a family of maximally knotless graphs with $3|V| - 6$ edges, which is the same number as for maximally planar graphs. On the other hand, in Theorem 11, we show that $\frac{3|V|}{2}$ is a lower bound for the number of edges in a maximally knotless graph.

I. Preliminary Theorems

Theorem 1: Let G be a maximally knotless graph. Then G is connected. If $|V| \geq 3$, then G has minimum degree ≥ 2.

Note that the maximally knotless graphs on 1 or 2 vertices, K_1 and K_2, (see Theorem 4) have minimum degree 0 and 1.

Proof.

Let G be a maximally knotless graph. Suppose G is not connected. This would mean that G has two or more components. In a knotless embedding of G, if we add an edge to connect two components, this would not result in a knot. But having added an edge to G without introducing a knot, we have contradicted G being maximally knotless. Thus, G must be connected.

Let G be a maximally knotless graph with $|V| \geq 3$.
G cannot contain a vertex with $\deg(v) = 0$ since G is a connected graph.
Suppose there is a vertex with $\deg(v) = 1$. Let u be the neighbor of v. We can connect v to any neighbors of u. For example, *Figure 3.1* shows a neighbor w.

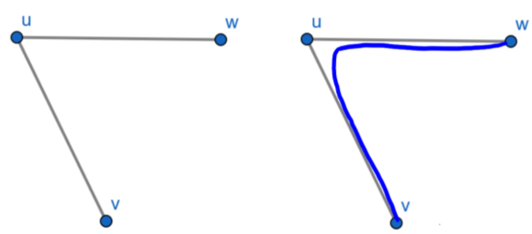


Figure 3.1: Theorem 1

In a knotless embedding of G, connect v to w by adding an edge which closely follows the path from v to u to w. Then the added edge does not introduce a knot and $G + vw$ is also knotless. This shows G is not maximally knotless, a contradiction. Thus, G has a minimum degree of at least 2. ■

Theorem 2: For $|V| = n \geq 7$, a maximally knotless graph has between 20 and $5n - 15$ edges.

Notice that the theorem does not hold for graphs with $1 \leq n < 7$ vertices. By Theorem 4 below, the only maximally knotless graphs with fewer than seven vertices are the complete graphs K_1 through K_6, with between zero and fifteen edges.

Proof.

The lower bound (20 edges) is developed from Mattman (2011) who shows that an intrinsically knotted graph has at least 21 edges. Suppose G is maximally knotless with $n \geq 7$ vertices. Then, since adding an edge gives an IK graph, G has at least 20 edges.

The upper bound ($5n - 15$ edges) is developed from Campbell et al. (2008) which states that a graph with $n \geq 7$ and at least $5n - 14$ edges is IK. By definition, an IK graph is not knotless, so a maximally knotless graph can have no more than $5n - 15$ edges. ■

Theorem 3: A 2-apex graph is maximally knotless if and only if it is maximally 2-apex.

Proof.
Let G be 2-apex.
Suppose G is maximally knotless. Then, by definition of maximally knotless, adding an edge ab (not already in G) would result in a graph that is IK. As stated in Chapter 2 (Knotless, Intrinsically Knotted and Maximally Knotless Graphs), we know that 2-apex graphs have a knotless embedding so $G + \{ab\}$ cannot be a 2-apex graph. Thus, G must be maximally 2-apex.

Suppose G is maximally 2-apex with n vertices.
We need to consider two cases: (1) when $n < 7$ and (2) when $n \geq 7$.

Case 1.
Let $n < 7$. Since K_1 through K_4 are planar (Chapter 2, Planarity), for $n < 7$ the maximally 2-apex graphs, G, are the graphs K_n. By Theorem 4 below each of these G is maximally knotless.

Case 2.
Let $n \geq 7$. This means that G has $5n - 15$ edges (Chapter 2, Apex graphs). Adding an edge ab to G would result in a graph with $5n - 14$ edges, which, by Campbell et al. (2008), means that $G + \{ab\}$ is IK. Since $G + \{ab\}$ is IK for any added edge ab, by definition G is maximally knotless.

So, if G is maximally 2-apex, then G is maximally knotless.
Thus, a 2-apex graph is maximally knotless if and only if it is maximally 2-apex.■

II. Classification of Maximally Knotless Graphs through Nine Vertices and Twenty Edges

Let $n = |V|$ denote the number of vertices in a graph G.

Theorem 4: For $n < 7$, the complete graph K_n is the only maximally knotless graph on n vertices. The complete graph with an edge removed, K_7^-, is the only maximally knotless graph on seven vertices. These seven graphs are the maximally knotless graphs with at most twenty edges.

Proof.
For $n \leq 7$, the complete graph K_7 is the only IK graph (see Chapter 2, The Heawood Family Graphs). This means that for $1 \leq n < 7$, K_n is maximally knotless since any minor of K_7 has a knotless embedding and no other edges can be added to the complete graphs K_1 through K_6. For graphs of exactly seven vertices, K_7^-, or K_7 with one edge removed, is the only maximally knotless graph. The graph K_7^- has a knotless embedding since it is a proper minor of K_7. The only edge that can be added to K_7– is the edge that would create K_7 which is IK. Thus, K_7^- is the only maximally knotless graph on seven vertices.

For the number of edges, note that Mattman (2011), shows every graph on 20 or fewer edges is 2-apex. By Theorem 3, to be maximally knotless, such a graph would need to be maximally 2-apex. If $n < 7$, the complete graphs K_1 to K_6 have $0 \leq n \leq 15$ edges.
If $n \geq 7$, this means $20 \geq 5n - 15$, since a maximally 2-apex graph has $5n - 15$ edges (Chapter 2, Apex Graphs). It follows that $n = 7$. So, a maximally knotless graph with at most 20 edges is one of the seven graphs with seven or fewer vertices.■

Theorem 5: There are two maximally knotless graphs on 8 vertices.

Proof.
Mattman (2011, Proposition 1.4) proves that all graphs with 8 vertices are either IK or 2-apex graphs. By definition, IK graphs are not maximally knotless. Thus, by Theorem 3, the only maximally knotless graphs on 8 vertices are the maximally 2-apex graphs. Mattman shows that that there are exactly two maximally 2-apex graphs, called G_1 and G_2. The complements of G_1 and G_2 are shown in *Figure 3.2* below.

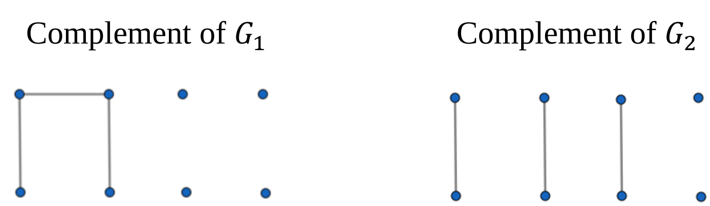


Figure 3.2: Complements of Graphs G_1 and G_2

Theorem 6: There are seven maximally knotless graphs on 9 vertices.

Proof.

Using the parameters of the number of edges in maximally knotless graphs from Theorem 2, a maximally knotless graph with 9 vertices will have between 20 and $5n - 15 = 30$ edges (since $n = 9$).

For graphs with 9 vertices and no more than 21 edges, Mattman (2011) (Propositions 1.6 and 1.7) proves that these graphs are either IK, the graph E_9, or 2-apex. By definition, IK graphs are not maximally knotless. We show the graph E_9 is maximally knotless in Theorem 8. By Theorem 3, 2-apex graphs are only maximally knotless if they are maximally 2-apex. As mentioned in Chapter 2 (Apex Graphs), a maximally 2-apex graph would have $5n - 15 = 30$ edges for $n = 9$ vertices. Therefore, E_9 is the only maximally knotless graph with at most 21 edges.

For graphs with 9 vertices with 22 to 30 edges, Morris (2008) classifies these graphs as either IK, 2-apex or one of 32 indeterminate graphs. Again, 2-apex graphs are only maximally knotless if they are maximally 2-apex (Theorem 3) and there are five of these. Their complements have been called House, Hat, Big-Y, Long-Y and Pentagon-Bar and can be seen in *Figure 3.3* below. Again, IK graphs are not maximally knotless which leaves only the 32 indeterminate graphs to explore.

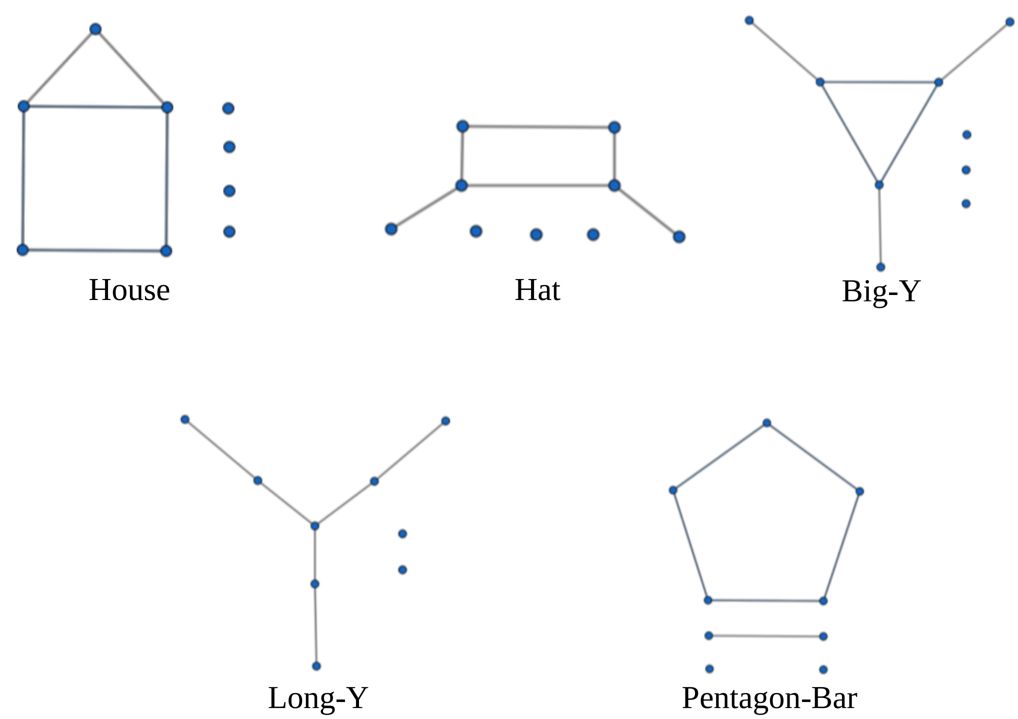


Figure 3.3: The Complements of Five Maximally 2-apex Graphs

In later work, Goldberg et al. (2014) showed that all but eight of the 32 indeterminate graphs are IK. The remaining eight are neither 2-apex nor IK. One of these is the graph E_9, which we show to be maximally knotless in Theorem 8. The other seven graphs have between 26 and 29 edges. We call the one with 29 edges $G_{(9,29)}$. In Theorem 7 below, we show this graph is maximally knotless. The remaining six graphs are obtained by deleting edges from $G_{(9,29)}$. Thus, they cannot be maximally knotless. In summary, the seven maximally knotless graphs on nine vertices are E_9, $G_{(9,29)}$, and the five maximally 2-apex graphs (House, Hat, Big-Y, Long-Y and Pentagon-Bar). ■

The Complement of $G_{(9,29)}$

The benefit of working with the complement of $G_{(9,29)}$ (see Chapter 2, Complement of a Graph) is that it is a simpler graph with fewer edges. It also reveals symmetry in the graph making it easier to identify isomorphisms when adding different types of edges . The complement of the graph $G_{(9,29)}$ can be derived as follows: beginning with $G_{(9,29)}$ (left), add in all additional edges to form a complete graph (middle) and remove all edges of $G_{(9,29)}$ to reveal the complement (right).

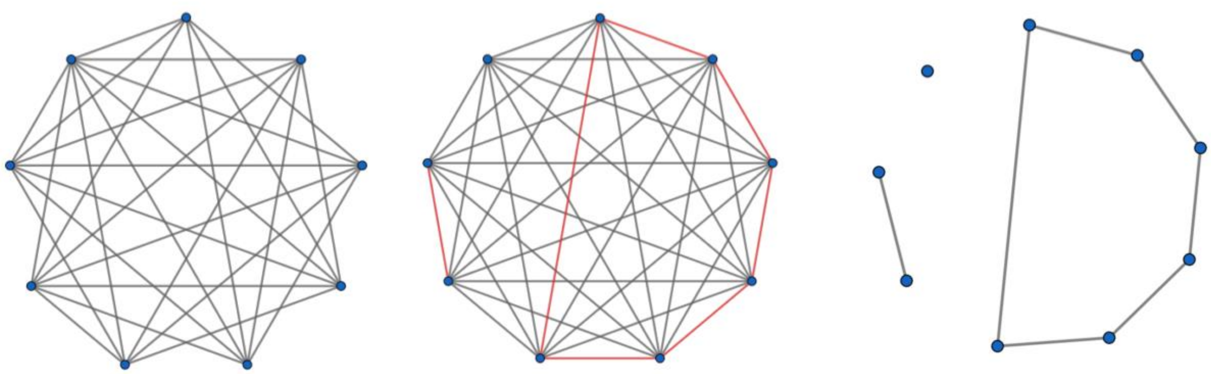


Figure 3.4: Constructing the Complement of $G_{(9,29)}$

Theorem 7: The graph $G_{(9,29)}$ is maximally knotless.

Proof.

Ryker (2013) gives a knotless embedding of the graph $G_{(9,29)}$, due to Naimi (see Mattman et al., 2017) shown in *Figure 3.5*.

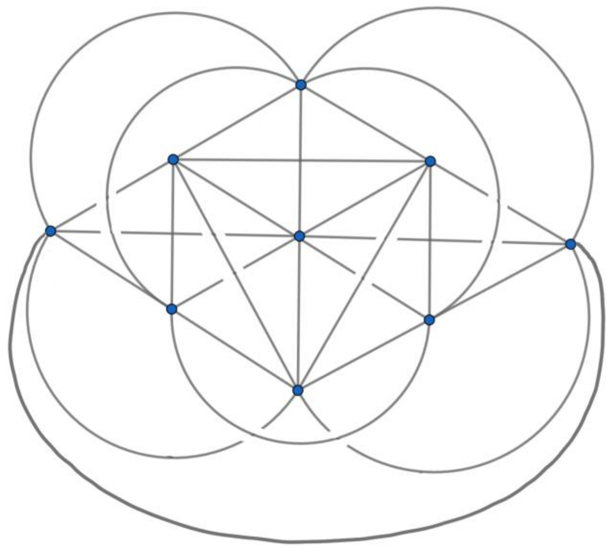


Figure 3.5: A Knotless Embedding of $G_{(9,29)}$

To show that $G_{(9,29)}$ is maximally knotless, we will use the complement as shown in *Figure 3.6*.

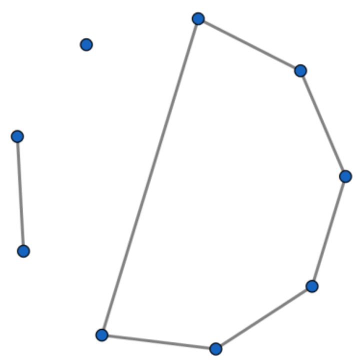
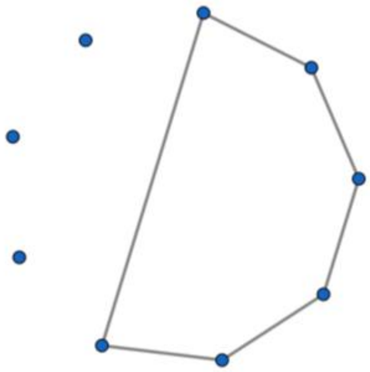


Figure 3.6: The Complement of $G_{(9,29)}$

Note that $G_{(9,29)}$ is one of the 32 indeterminate graphs from Morris (2008). Morris showed that, aside from those 32, every other connected graph on nine vertices is either IK or 2-apex. As mentioned in the proof of the previous theorem these 32 graphs each have at most 29 edges. Thus, adding an edge to $G_{(9,29)}$, the resulting graph on 30 edges cannot be one of the other 31 indeterminate graphs. So, by Morris's classification, $G_{(9,29)}$ with an additional edge is either 2-apex or IK.

By adding an edge to $G_{(9,29)}$, this will remove an edge from the complement. This can be done in two ways (see *Figure 3.7*). The complement of $G_{(9,29)} + e$ will result in a hexagon and three vertices with $\deg(v) = 0$. The complement of $G_{(9,29)} + f$ will result in a path containing 6 vertices and 5 edges, the complete graph K_2 and one vertex with $\deg(v) = 0$.


$G_{(9,29)} + e$

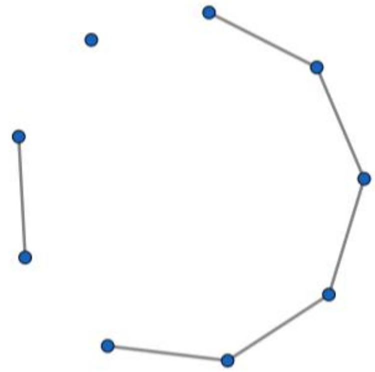

$G_{(9,29)} + f$

Figure 3.7: Adding Edges to $G_{(9,29)}$

Neither of these cases result in a graph isomorphic to the complements of the five maximally 2-apex graphs with 30 edges from Theorem 6. So, $G_{(9,29)}$ with an addition edge is not maximally 2-apex. This means it is not 2-apex. Since Morris showed a graph on 30 edges is either 2-apex or IK, the graph $G_{(9,29)}$ with an additional edge must be IK. Therefore, $G_{(9,29)}$ is maximally knotless. ■

Lemma 1: This embedding of E_9 is knotless.

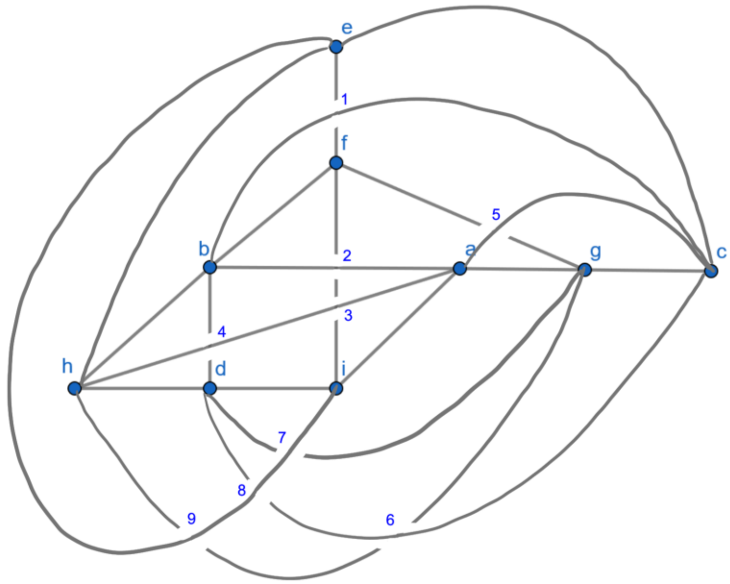


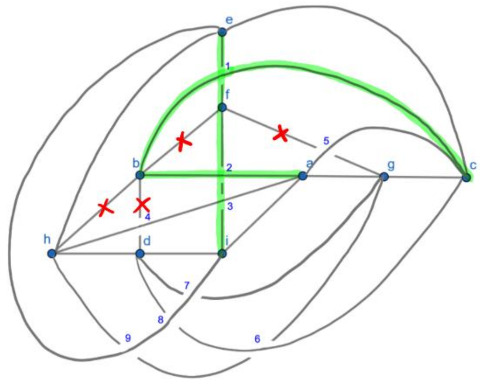
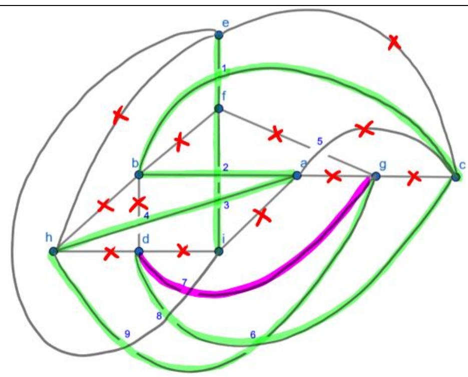
Figure 3.8: A Knotless Embedding of E_9

Proof.

For the above embedding of E_9 to be knotless it cannot contain any non-trivially knotted cycles. As discussed in Chapter 2 (Knots, Reidemeister Moves, and Equivalent Knots), the following must be true for a projection of a non-trivial knot: (1) it has at least three crossings and (2) if the projection includes consecutive non-alternating crossings, there must be at least four crossings. We will examine, one by one, the ways that we might have three or more crossings in a cycle. We conclude that any such cycle is in fact an unknot. This shows that the embedding of E_9 is knotless.

We first focus on four places in the embedding where consecutive non-alternating crossings could appear. Option 1 includes crossing combinations containing at least one of the following: 1&2, 2&3, or 1&3. Option 2 includes crossing combinations containing at least one of the following: 7&8, 8&9, or 7&9. Option 3 includes crossing combinations containing 3&4. Option 4 includes crossing combinations containing 6&9. For example, Table 3.1 gives the argument for cycles that include crossings 1&2. In the table, the edges that form these crossings are highlighted in green in each diagram. The red x mark denotes edges that cannot be used since the adjacent vertices already have degree 2. See Appendix A for the remaining cases.

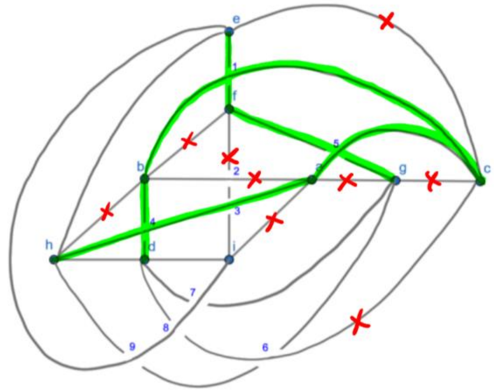
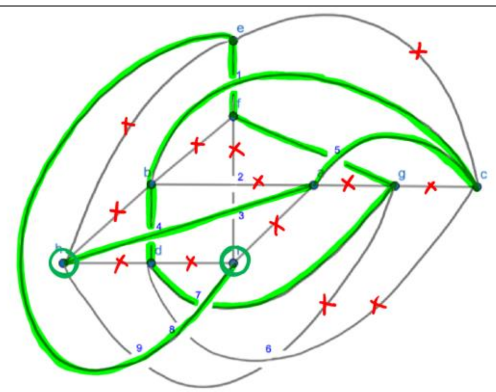
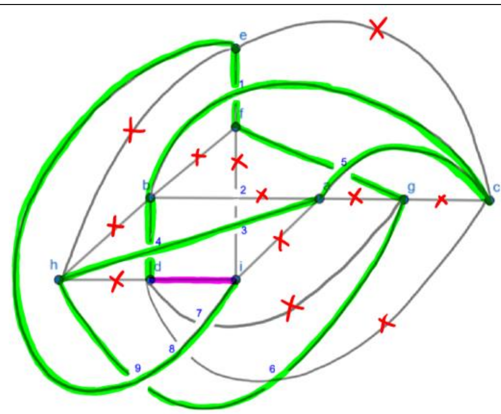
Table 3.1: Exploring Crossing 1&2

Crossings	Diagram	Explanation
1&2		When including crossings 1&2, since they are non-alternating at least 2 more crossings are needed in order to create a non-trivial knot. Crossing 4 is not possible since $deg(b) = 2$ causing edge bd to be unavailable. Crossing 5 is not possible since $deg(f) = 2$ causing edge fg to be unavailable. Crossings 7, 8, and 9 require adding edge ei, which creates a cycle $\{efie\}$. There is no knot in these cases. This leaves {1,2,3,6} as the only way to get to four crossings.
{1,2,3,6}		Once edges cd and gh are added to create crossing 6, this leaves only one available edge at g. Add edge dg (pink). Since $deg(d) = 2$, edges dh and di become unavailable leaving i with only one possible edge, ei. However, adding ei creates cycle $\{efie\}$. Thus, {1,2,3,6} will not create a knot.
Therefore, crossings 1&2 with two additional crossings will not create a non-trivial knot.		

After eliminating these four options, there are 36 triples left to check:
$\{1,4,5\}, \{1,4,6\}, \{1,4,7\}, \{1,4,8\}, \{1,4,9\}, \{1,5,6\}, \{1,5,7\}, \{1,5,8\}, \{1,5,9\}, \{1,6,7\}, \{1,6,8\}, \{2,4,5\}, \{2,4,6\}, \{2,4,7\}, \{2,4,8\}, \{2,4,9\}, \{2,5,6\}, \{2,5,7\}, \{2,5,8\}, \{2,5,9\}, \{2,6,7\}, \{2,6,8\}, \{3,5,6\}, \{3,5,7\}, \{3,5,8\}, \{3,5,9\}, \{3,6,7\}, \{3,6,8\}, \{4,5,6\}, \{4,5,7\}, \{4,5,8\}, \{4,5,9\}, \{4,6,7\}, \{4,6,8\}, \{5,6,7\}, \{5,6,8\}.$

Table 3.2 is an exploration of cycles for the first triple listed above, {1,4,5}. This exploration examines all cycles that include the three crossings at 1, 4, and 5. For an exploration of the remaining 35 triples please see Appendix B. ■

Table 3.2: Exploration of Triple $\{1, 4, 5\}$

Triple	Diagram	Explanation
{1,4,5}		Since crossings 1 and 5 are consecutive non-alternating crossings at least one other crossing is required for a non-trivial knot. $\{1, 2, 4, 5\}$: Already included in Appendix A. $\{1, 3, 4, 5\}$: Included in Appendix A. $\{1, 4, 5, 6\}$: Edge cd is unavailable since $\deg(c) = 2$. $\{1, 4, 5, 7\}$: See *Figure D1 below.* $\{1, 4, 5, 8\}$: Edge cd is unavailable since $\deg(c) = 2$. $\{1, 4, 5, 9\}$: See *Figure D2 below.*
Figure D1		$\{1, 4, 5, 7\}$: After adding edges dg and ei to create crossing *7*, no other edges are available to create a knot since $\deg(d) = 2$, $\deg(e) = 2$ and $\deg(g) = 2$ leaving $\deg(h) = 1$ and $\deg(i) = 1$. There is no cycle with theses four crossings.
Figure D2		$\{1, 4, 5, 9\}$: After adding edges gh and ei to create crossing 9, only one edge is available to connect vertices d and i (highlighted in pink).This creates a cycle in the graph which, through the use of Reidemeister moves, is revealed to be an unknot.
Cycles with crossings {1,4,5} will not result in a non-trivial knot.		

The Complement of E_9

Similar to working with the complement of $G_{(9,29)}$, we can use the symmetry of the complement of E_9 to identify isomorphisms when adding different types of edges . The complement of the graph E_9 can be derived as follows: beginning with E_9 (left), add in all additional edges to form a complete graph (middle) and remove all edges of E_9 to reveal the complement (right).

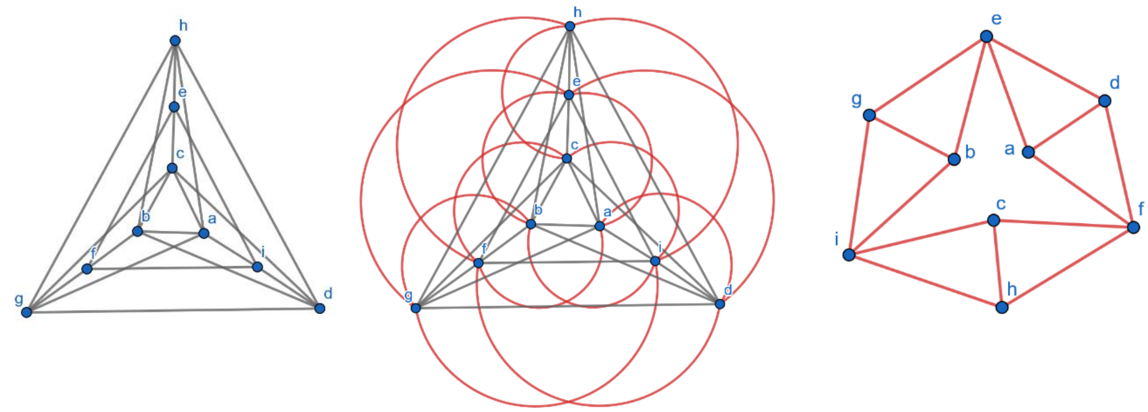


Figure 3.9: Constructing the Complement of E_9

Theorem 8: - The graph E_9 is maximally knotless.

Proof.
For E_9 to be maximally knotless, the following must hold: (1) E_9 has a knotless embedding and (2) by adding an edge to E_9 the graph becomes intrinsically knotted (IK).

By Lemma 1, the given embedding of E_9 is knotless.

Now consider adding an edge to E_9. Using the symmetry of the complement graph, there are two ways to add an edge.

First, consider edge e, an edge added to E_9 such that it connects one vertex with degree 4 and one vertex of degree 5. Below is $E_9 + e$ (left) and its complement (right).

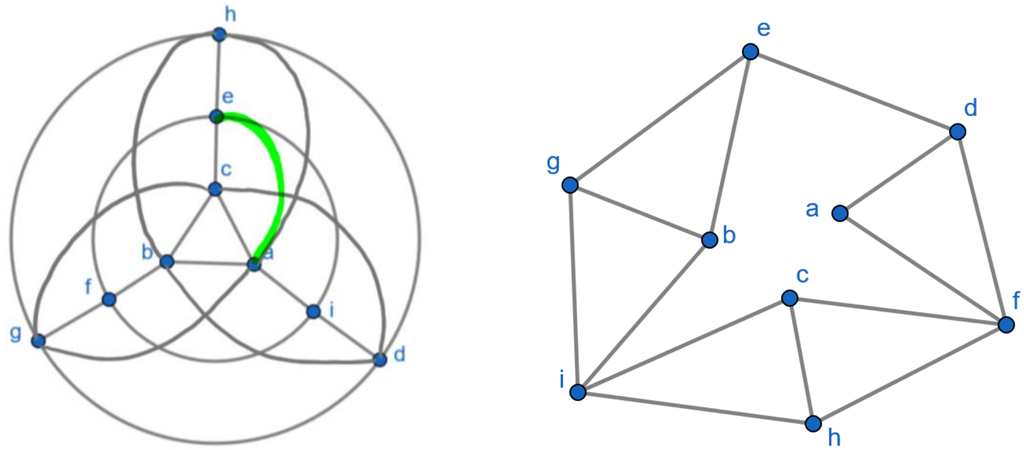


Figure 3.10: $E_9 + e$ and Its Complement

In Goldberg et al. (2014) the graph $E_9 + e$ is proven to be intrinsically knotted.

Next, consider edge f which connects two vertices with degree 5. Below is $E_9 + f$ (left) and its complement (right).

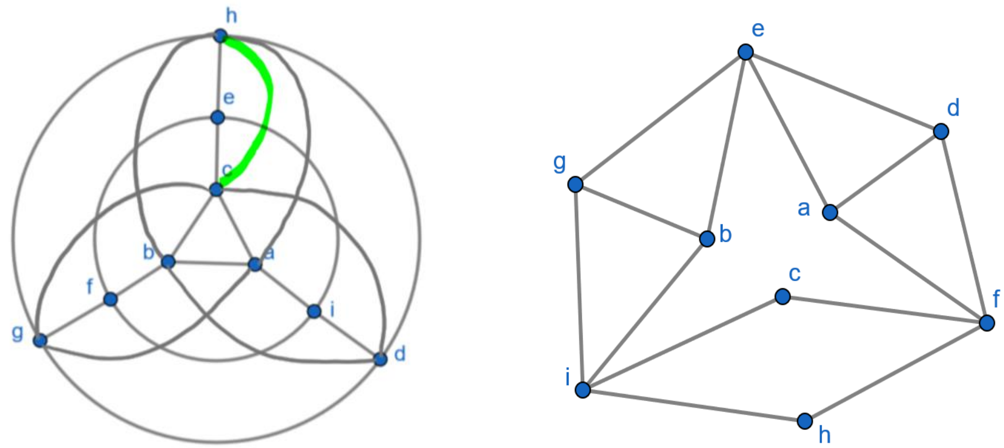


Figure 3.11: $E_9 + f$ and Its Complement

In the graph above ($E_9 + f$) remove the edge fi. The graph $E_9 + f - \{fi\}$ is the graph F_9 in the Heawood Family, which is a known IK graph, as mentioned in Chapter 2 (Heawood Family). Since $E_9 + f$ has an IK minor, it is also IK (see Chapter 2, Minor of a Graph). ■

III. Number of Edges in a Maximally Knotless Graph

The next three theorems and their proofs very closely follow the work of Max Aires (2019).

Theorem 9: There is a maximally knotless graph with 15 vertices and 39 edges

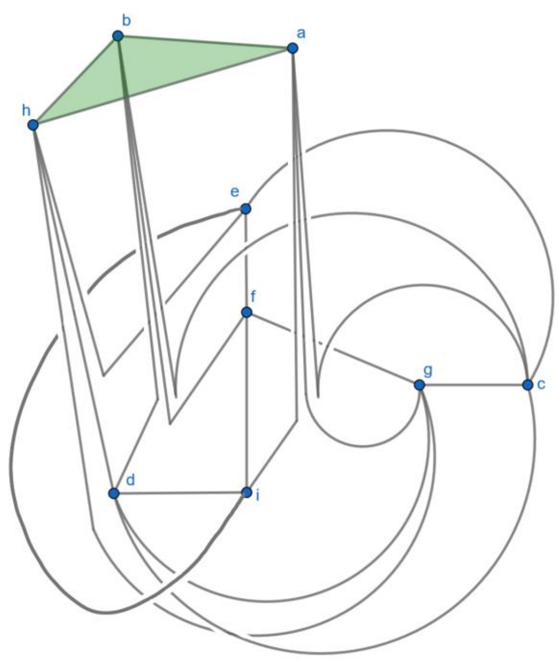


Figure 3.12: Suspended Triangle abh

Proof.
In *Figure 3.12* above, the knotless embedding of E_9 in Theorem 8 has been modified by topologically moving the vertices and edges of the triangle abh to lie on a plane above the rest of the graph, $G\backslash\{a, b, h\}$, which lies near an entirely different plane below.

Consider two copies of graph E_9 where G_1 and G_2 are both the knotless embeddings of E_9 as seen in *Figure 3.12*. We will label the vertices of G_1 as $a_1, b_1, \dots, i_1$ and G_2 as $a_2, b_2, \dots, i_2$.

Combine the graphs G_1 and G_2 such that they share a common triangle (a_1 coincides with a_2, b_1 coincides with b_2 and h_1 coincides with h_2) to form a new graph, H, with a total of 15 vertices and 39 edges. See *Figure 3.13* below.

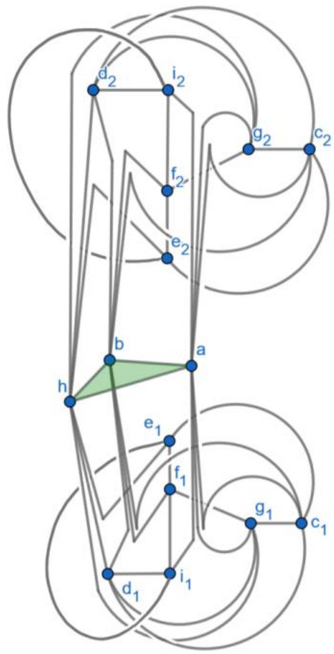


Figure 3.13: Two Copies of G

First, note that a cycle completely contained in a single G_i is an unknot by Theorem 8.

Next, notice that if a cycle C is not completely contained in either G_1 or G_2, then it must pass through the triangle connecting the graphs twice: once travelling from G_2 to G_1 and then back from G_1 to G_2. Then we can write C as the knot sum of a cycle C_1 in G_1 and another C_2 in G_2. By the knot sum theorem (see Chapter 2, Knot Sum Theorem), the sum of the two unknots is again an unknot.

Hence, the embedding of the graph H formed by combining $G_1$and G_2 is again knotless.

Suppose H were not maximally knotless and that we could add some edge uv to it. Note that since each copy of E_9 is maximally knotless, uv cannot be entirely contained within one copy of E_9 and thus u and v must be in different copies. Let's say u is in G_1 and v in G_2.

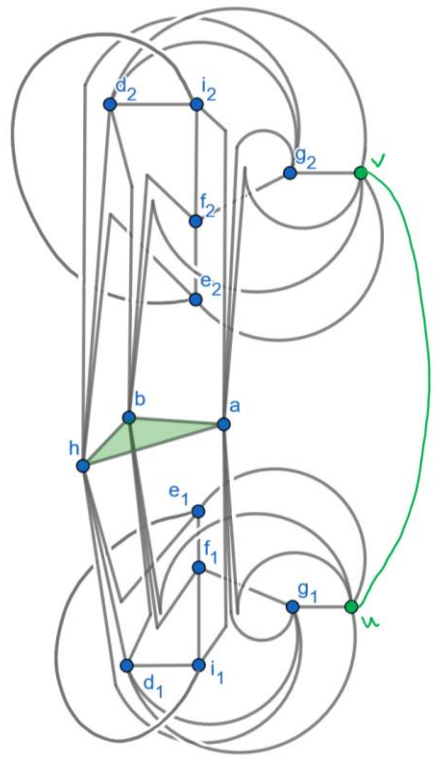


Figure 3.14: Two Copies of $G + \{uv\}$

Use edge contractions (see Chapter 2, Edge Contraction) to contract G_2 to a single point.

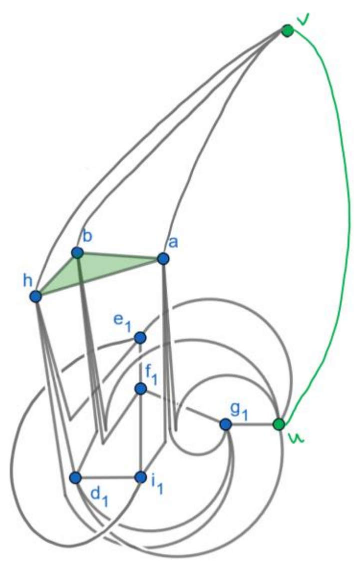


Figure 3.15: Contracting G_2

Next, contract the edge uv. Observe that no vertex of G_1 is adjacent to all three vertices a, b and h. However, contracting uv causes u to be adjacent to all three vertices in the common triangle and thus creates an embedding of E_9 with at least one additional edge. Since it was assumed that $H + uv$ was a knotless embedding, by construction there exists a knotless embedding of a graph formed by adding at least one edge to E_9, which is impossible.

Therefore, H must also be maximally knotless. ■

Theorem 10: There exist maximally knotless graphs with n vertices and $m < 3n$ edges for arbitrarily large n.

Specifically, for each $k > 1$ we will construct a graph with $n = 3 + 6k$ and $m = 3n - 6$.

Proof.

Let k > 1 and let H be the graph formed by combining k copies of E_9 (see *Figure 3.12*) along a common triangle.

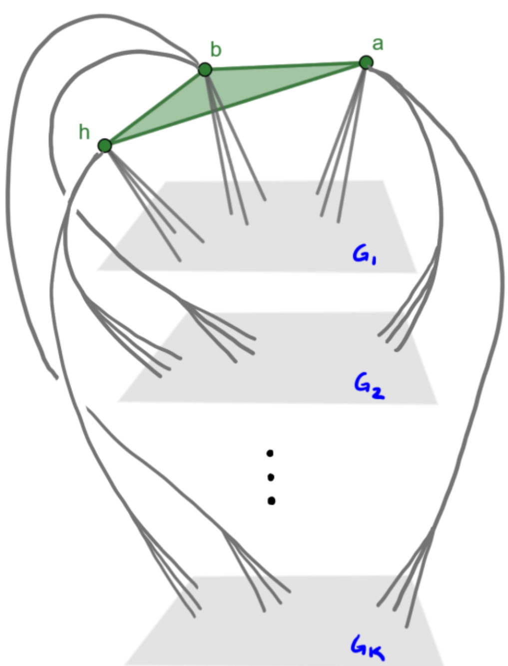


Figure 3.16: k Copies of G

First, note that a cycle completely contained in a single G_i is an unknot by Theorem 8.

Next, notice that if a cycle C is not completely contained in a single G_i, then it must pass through the triangle connecting the graphs twice or three times. If the cycle is in two copies of E_9, then, as in the previous theorem, it is the sum of two unknots and itself an unknot. Suppose the cycle passes through the triangle three times: once travelling from G_1 to G_2, then from G_2 to G_3 and a third time from G_3 back to G_1.

Then we can write C as the knot sum of a cycle C_1 in G_1, another C_2 in G_2, and a third cycle C_3 in G_3. By the knot sum theorem of Chapter 2, the sum of the three unknots is again an unknot. Hence, the embedding of graph H formed by combining k copies of E_9 is again knotless.

Suppose H were not maximally knotless and that we could add some edge uv to it. Note that since each copy of E_9 is maximally knotless, uv cannot be entirely contained within one copy of E_9 and thus u and v must be in different copies. See *Figure 3.17* below.

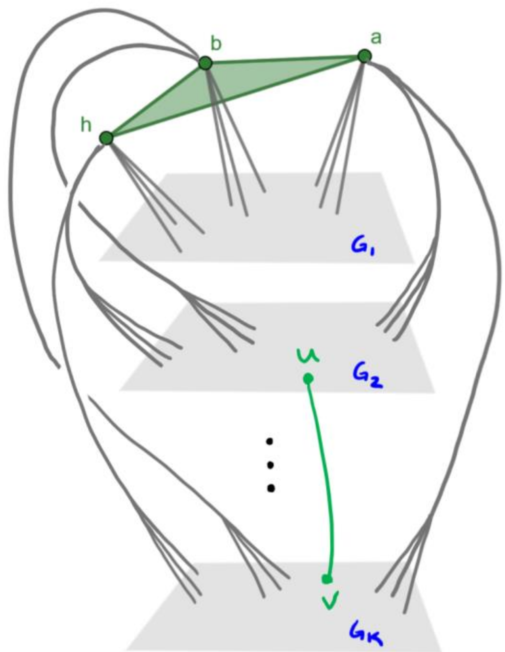


Figure 3.17: k Copies of $G + \{uv\}$

Contract the copy of E_9, call it G_i, containing v to a single point. In other words, the G_i copy of E_9 contains the vertices $\{a, b, c_i, d_i, e_{i,} f_i, g_i$ and $h\}$ where a, b and h are in the common triangle and the remaining vertices are in a plane below. We can contract all vertices in the plane to a single point v that is now adjacent to each vertex a, b and h.

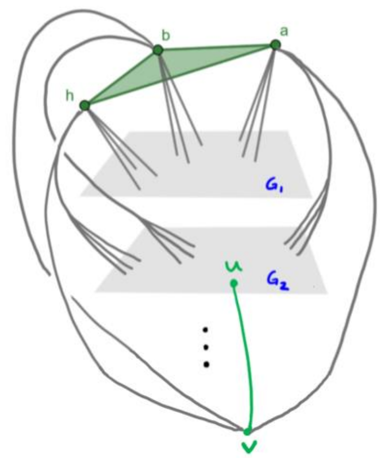


Figure 3.18: Contracting G_i

Next, contract the edge uv. Observe that no vertex in E_9 is adjacent to all three vertices a, b and h in the common triangle. However, contracting uv causes u to be adjacent to all three vertices in the common triangle and thus creates an embedding of a graph formed by adding at least one edge to E_9. Since it was assumed that $H + uv$ is a knotless embedding, by construction there exists a knotless embedding of E_9 with at least one added edge, which is impossible. Therefore, H must also be maximally knotless.

Notice, H has $n = 3 + 6k$ vertices and $m = 3 + 18k$ edges. Thus, for any k > 1, we have constructed a maximally knotless graph with $n = 3 + 6k$ vertices and $m = 3 + 18\left(\frac{n-3}{6}\right) = 3n - 6$. ■

Finally, we shall prove a lower bound on the number of edges in a maximally knotless graph.

Theorem 11: Let G be a maximally knotless graph with $n \geq 4$ vertices and m edges. Then, $m \geq \frac{3n}{2}$.

Notice that the maximally knotless graphs for $n < 4$ vertices (see Theorem 4) do not satisfy the Theorem.

K_1 is the only maximally knotless graph for $n = 1$ vertices and has $m = 0$ edges. So, $m < \frac{3n}{2}$.

K_2 is the only maximally knotless graph for $n = 2$ vertices and has $m = 1$ edges. So, $m < \frac{3n}{2}$.

K_3 is the only maximally knotless graph for $n = 3$ vertices and has $m = 3$ edges. So, $m < \frac{3n}{2}$.

While this proof closely follows the work of Aires (2019), the conclusion of this proof $\left(m \geq \frac{3n}{2}\right)$ is weaker. Aires (2019) instead deduces $m \geq 2n$ for maximally linkless graphs.

Proof.

The only maximally knotless graph on 4 vertices is K_4 for which $m = 6$ and $m \geq \frac{3n}{2}$.

Suppose there exists a maximally knotless graph with $m < \frac{3n}{2}$ edges and let H be such a graph with the minimum number of vertices n where $n > 4$. By the handshake lemma, the total degree of H would be $\sum \deg(v_i) = 2|E| = 2m < 3n$. Thus, the average degree is less than 3 so there is a vertex v contained in H with $\deg(v) \leq 2$.

By Theorem 1, the minimum degree of H is at least 2. So, $\deg(v)$ must be exactly 2. Then the two neighbors of v must be connected, as we can otherwise add an edge closely following the path of the edges to v without creating a knot. See *Figure 3.9* below.

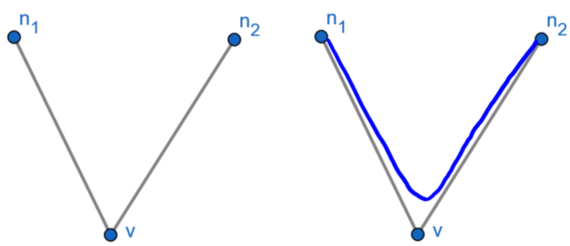


Figure 3.19: Theorem 11

We can show that $H - v$ is maximally knotless. To show $H - v$ is maximally knotless, this requires two things: (1) $H - v$ is knotless and (2) adding an edge makes the graph IK.

Since H has a knotless embedding, removing a vertex v and its adjacent edges would result in a knotless embedding of $H - v$.

Suppose instead that $H - v$ is not maximally knotless. Then there is a knotless embedding even after adding an edge $\{ab\}$. In other words, suppose $H - v + \{ab\}$ has a knotless embedding. As mentioned earlier, (see Figure 3.18), H includes the edge $n_1 n_2$ between the neighbors of v. In the knotless embedding of $H - v + \{ab\}$, we can add v to the graph without introducting a knot by connecting v to its two neighbors n_1 and n_2 with edges that follow the edge $n_1 n_2$. This means $H + \{ab\}$ also has a knotless embedding. This contradicts H being maximally knotless. Thus, $H - v$ must be maximally knotless.

So, $H - v$ is maximally knotless and has $m - 2 < \frac{3(n-1)}{2}$ edges, contradicting the minimality of H. ■

CHAPTER 4

CONCLUSION AND FURTHER RESEARCH

In this Chapter, we use the Theorems we just proved as a springboard to some topics for further research. We will explore the boundaries of different aspects of maximally knotless graphs suggested by our theorems.

Theorem 1 states that the minimum degree for a maximally knotless graph is at least 2. However, aside from the complete graphs K_n with $n < 5$, all of our examples have a minimum degree of at least four. The graphs K_5 and E_9 have minimum degree four. Is it possible to have a maximally knotless graph with a minimum degree less than four (and more than four vertices)?

Theorem 1 also discusses the idea of connectivity of maximally knotless graphs. We know that maximally knotless graphs must be connected but what is the smallest connectivity? Is connectivity one possible for a maximally knotless graph?

Theorems 4 through 8 discuss maximally knotless graphs with $1 \leq n \leq 9$ vertices and at most twenty edges. What are the maximally knotless graphs on 10 vertices? We know that the maximally 2-apex graphs with 5(10) – 15 edges are maximally knotless, but are there more examples that are not 2-apex, like E_9 and $G_{(9,29)}$? What about graphs with more than 10 vertices? In Theorem 9 we have proven a graph with 15 vertices and 39 edges to be maximally knotless. Are there others? For edges, we have E_9 with 21 edges and the maximally 2-apex eight vertex graphs G_1 and G_2 with 25 edges. Are there other examples with 21 edges? Are there any maximally knotless graphs with $22 \leq |E| \leq 24$?

Another area to explore stems from Theorems 10 and 11. Our results for a lower bound on the number of edges for maximally knotless graphs is $m < 3n$ (Theorem 10) and $m \geq \frac{3n}{2}$ (Theorem 11). How can we close the gap between $3n$ and $\frac{3n}{2}$? Aires (2019) found a lower bound of $m \geq 2n$ for maximally linkless graphs. Should maximally knotless graphs also have a lower bound closer to $2n$?

The topic of maximally knotless graphs is still very new to the literature. This thesis is a first attempt at exploring these types of graph but there is more research that can be done to answer the questions listed above and others. We hope that this thesis will inspire others to continue research in this area of mathematics.

REFERENCES

Adams, C. C. (2004). *The knot book: An elementary introduction to the mathematical theory of knots.* Revised reprint of the 1994 original. American Mathematical Society, Providence, RI, 2004. xiv+307 pp. ISBN: 0-8218-3678-1

Aires, M. (2019, November 19). On the Number of Edges in Maximally Linkless Graphs. Retrieved January 8, 2020, from https://arxiv.org/abs/1911.08552

Blain, P., Bowlin, G., Fleming, T., Foisy, J., Hendricks, J., Lacombe, J. (2007). *Some results on intrinsically knotted graphs*. J. Knot Theory Ramifications 16, no. 6, 749--760.

Campbell, J., Mattman, T. W., Ottman, R., Pyzer, J., Rodrigues, M., Williams, S. (2008). *Intrinsic knotting and linking of almost complete graphs.* Kobe J. Math. 25, no. 1-2, 39-58.

Conway, J., Gordon, C. (1983). *Knots and links in spatial graphs.* J. Graph Theory 7, 445-453.

Dehkordi, H. R., Farr, G. (2019, July 23). Non-separating planar graphs. Retrieved January 8, 2020, from https://arxiv.org/abs/1907.09817

Goldberg, N., Mattman, T. W., Naimi, R. (2014). *Many, many more intrinsically knotted graphs.* Algebr. Geom. Topol. 14 , no. 3, 1801-1823.

Hanaki, R., Nikkuni, R., Taniyama, K., Yamazaki, A. (2011) *On intrinsically knotted or completely 3-linked graphs.* Pacific J. Math. 252, no. 2, 407-425.

Huck, S. (2010). *Intrinsic Knotting of Bipartite Graphs.* [Unpublished Degree Masters Thesis]. California State University, Chico.

Jørgensen, S. (1989). *Some maximal graphs that are not contractible to K_6*. Aalborg Universitetscenter. Institute for Elektroniske Systemer – Rapport, no. 1989 R 89-28.

Kohara, T., Suzuki, S. (1992). *Some remarks on knots and links in spatial graphs*. Knots 90 (Osaka, 1990), 435-445, de Gruyter, Berlin.

Kuratowski, K. (1930). *Sur le problème des courbes gauche en topologie.* Fund. Math. 15 271-283.

Marcus, D. A. (2008). *Graph theory: A problem oriented approach.* MAA Textbooks. Mathematical Association of America, Washington, DC,. xvi+205 pp. ISBN: 978-0-88385-753-3.

Maharry, J. (1998). *A splitter for graphs with no Petersen family graph minor*. Journal of Combinatorial Theory, Series B, 72, 136-139.

Mattman, T. W. (2011). *Graphs of 20 edges are 2-apex, hence unknotted.* Algebr. Geom. Topol. 11, no. 2, 691-718.

Mattman, T. W., Morris, C., Ryker, J. (2017). *Order nine MMIK graphs.* Knots, links, spatial graphs, and algebraic invariants, 103-124, Contemp. Math., 689, Amer. Math. Soc., Providence, RI.

Morris, Chris. (2008). *A Classification Of All Connected Graphs On Seven, Eight, And Nine Vertices With Respect To The Property Of Intrinsic Knotting*. [Unpublished Degree Masters Thesis]. California State University, Chico.

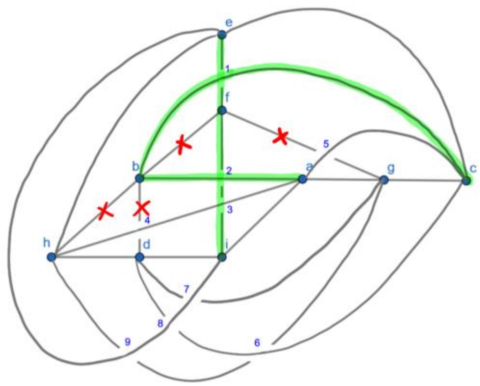
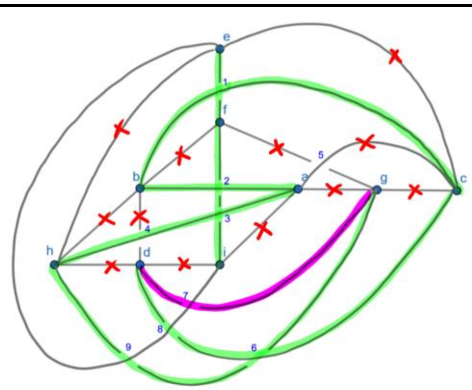
Naimi, R., Pavelescu, A., Pavelescu, E. (2020). New bounds on maximally linkless graphs. Retrieved July 28, 2020, from https://arxiv.org/abs/2007.10522

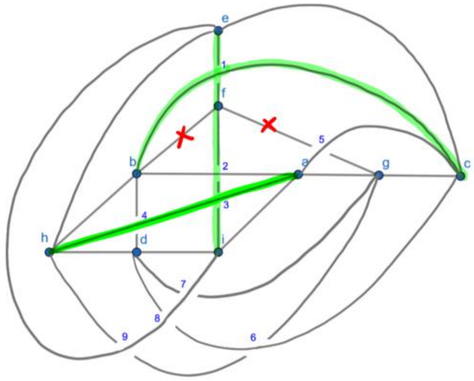
Ozawa, M., Tsutsumi, Y. (2007). *Primitive spatial graphs and graph minors.* Rev. Math. Complut. 20, 391-406

Robertson, N., Seymour, P., Thomas, R. (1993). *Linkless embedding of graphs in 3-space.* Bulletin of the Amer. Math. Soc. 28, 84-89.

Ryker, Jody (2013). *Knotting of Graphs on Nine Vertices and 28 or More Edges.* [Unpublished Degree Masters Thesis]. California State University, Chico.

Sachs, H. (1984). *On spatial representations of finite graphs.* Finite and infinite sets, Vol. I, II (Eger, 1981), 649-662, Colloq. Math. Soc. János Bolyai, 37, North-Holland, Amsterdam.

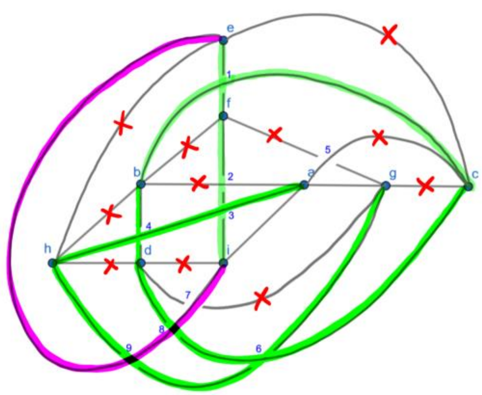
Crossings	Diagram	Explanation
1&2		When including crossings 1&2, since they are non-alternating at least 2 more crossings are needed in order to create a non-trivial knot. Crossing 4 is not possible since deg(b)=2 causing edge bd to be unavailable. Crossing 5 is not possible since deg(f)=2 causing edge fg to be unavailable. Crossings 7, 8, and 9 require adding edge ei, which creates a cycle {e,f,i,e}. There is no knot in these cases. This leaves 1236 as the only way to get to four crossings.
{1,2,3,6}		Once edges cd and gh are added to create crossing 6, this leaves only one available edge at g. Add edge dg (pink). Since deg(d)=2, edges dh and di become unavailable leaving i with only one possible edge, ei. However, adding ei creates cycle {e,f,i,e}. Thus, {1,2,3,6} will not create a knot.
Therefore, crossings 1&2 with two additional crossings will not create a non-trivial knot.		

<table>
<tr>
<td>1&3</td>
<td>
</td>
<td>When including crossings 1&3, since they are non-alternating at least 2 more crossings are needed in order to create a non-trivial knot.

Crossing 5 is not possible since deg(f)=2 causing edge fg to be unavailable.

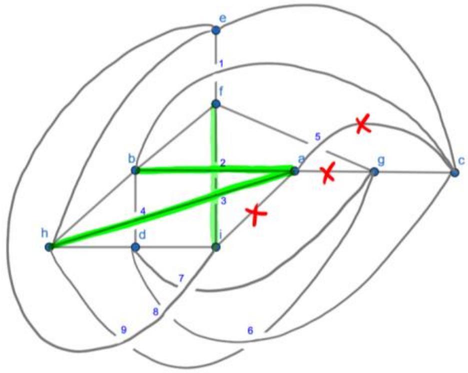
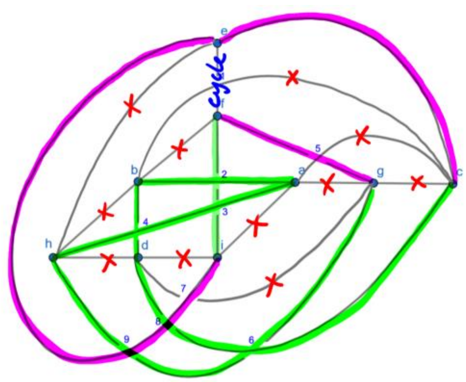
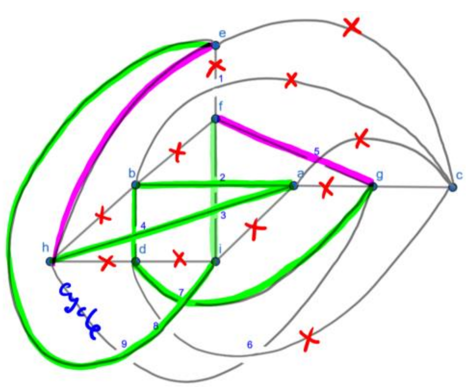
Note {1,2,3,n} is explored above so we don’t need to consider adding crossing 2.

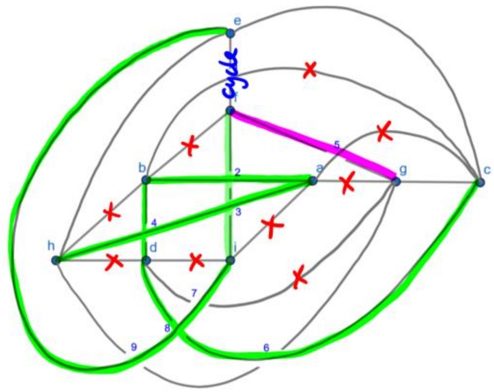
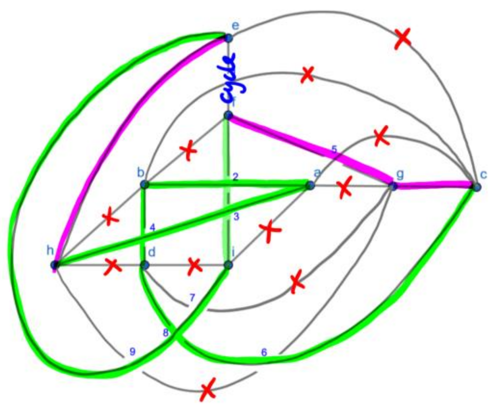
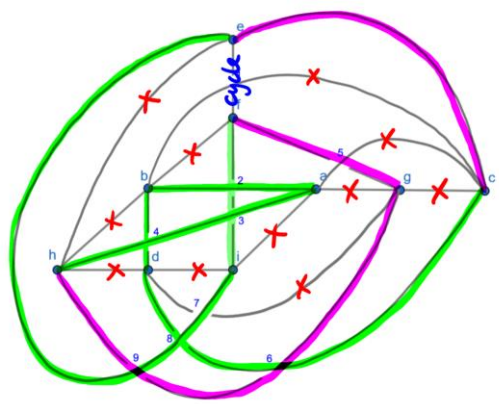
Crossings 7, 8 and 9 require adding edge ei, which makes the cycle {e,f,i,e} meaning there is no knot.

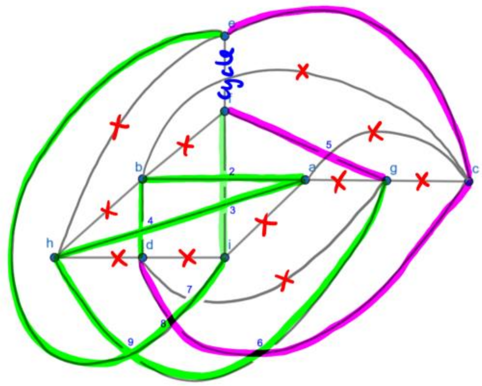
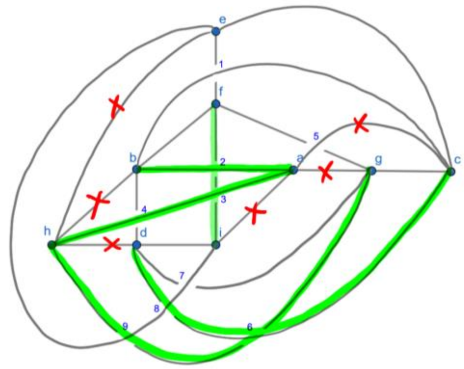
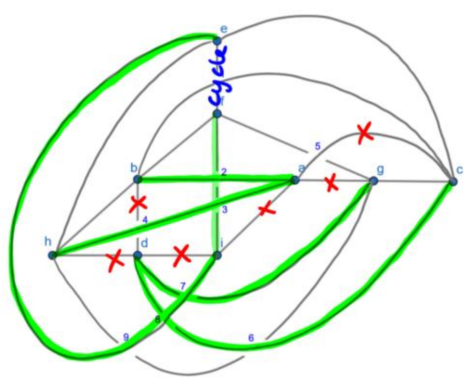
Thus, the only way to get to four crossings is {1,3,4,6}.</td>
</tr>
<tr>
<td>{1,3,4,6}</td>
<td>
</td>
<td>There is only one edge available at e since deg(c)=2 and deg(h)=2.

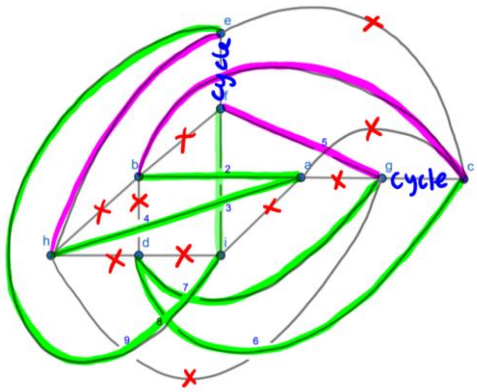
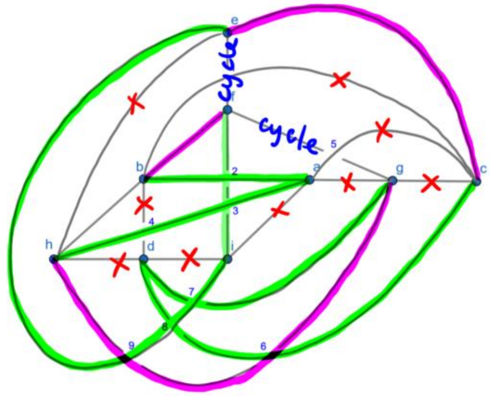
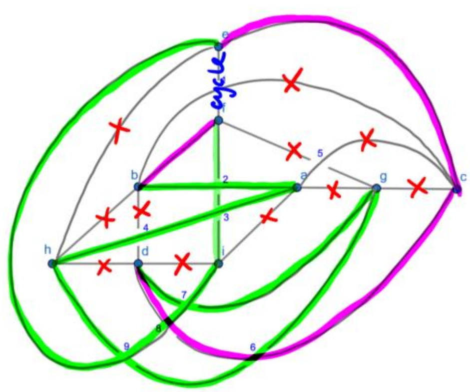
Adding edge ei (pink) creates cycle {e,f,i,e}.

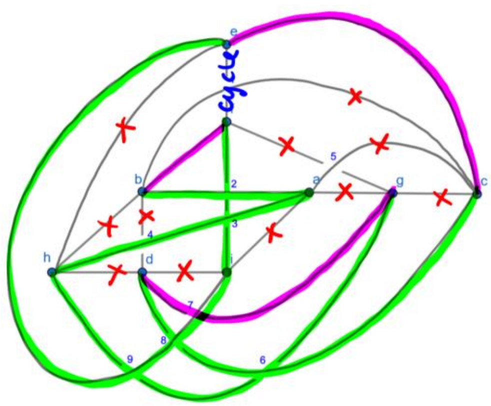
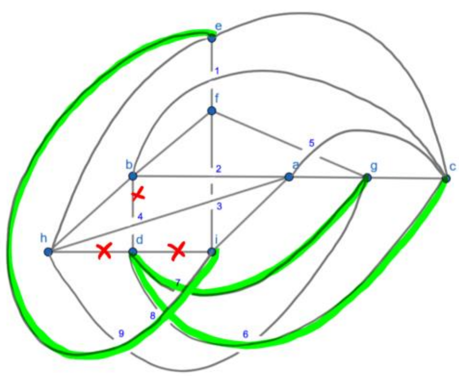
Thus, {1,3,4,6} will not create a knot.</td>
</tr>
<tr>
<td colspan="3">Therefore, crossings 1&3 with two additional crossings will not create a non-trivial knot.</td>
</tr>
</table>

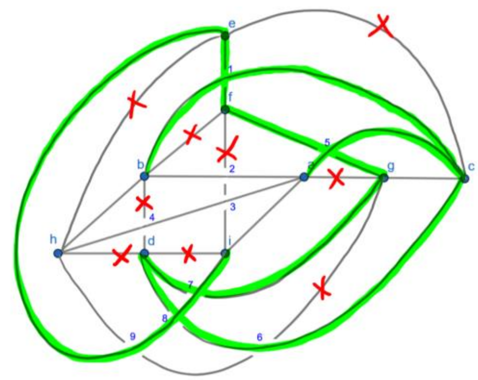
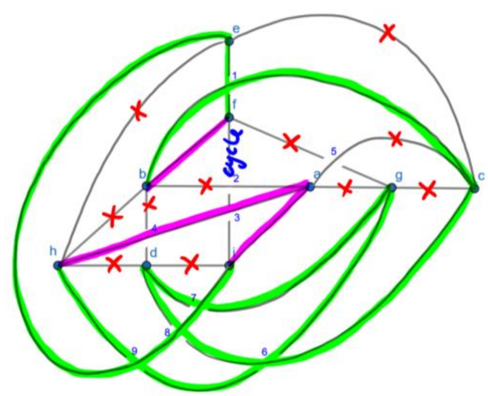
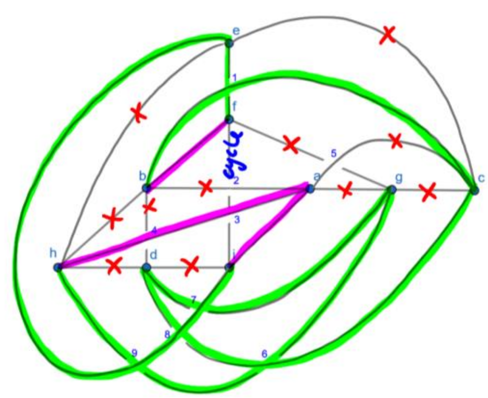
2&3		When including crossings 2&3, since they are non-alternating at least 2 more crossings are needed in order to create a non-trivial knot. Crossing 5 is not possible since deg(a)=2 causing edge fg to be unavailable. Note {1,2,3,n} is explored above so we don't need to consider adding crossing 1. There are 5C2 = 10 ways to add 2 more crossings: {2,3,4,6}, {2,3,4,7}, {2,3,4,8}, {2,3,4,9}, {2,3,6,7}, {2,3,6,8}, {2,3,6,9}, {2,3,7,8}, {2,3,7,9}, {2,3,8,9}
{2,3,4,6}		Since i has one available edge, add ei (pink). Since ef would create a cycle (blue), there is only one available edge at e. Add ce (pink). This also leaves only one available edge at f. Add fg (pink). This results in a trivial knot.
{2,3,4,7}		Since gh would create a cycle, this leaves only one available edge at h. Add eh (pink). This eliminates edge ef, leaving only one edge available. Add fg (pink). This results in a trivial knot.

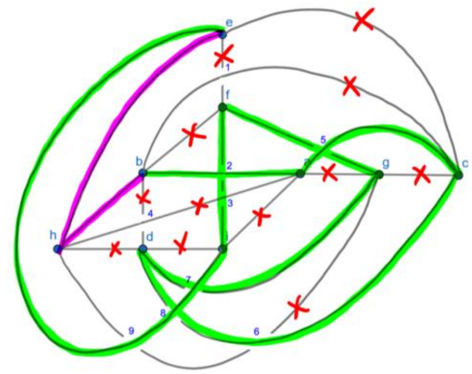
{2,3,4,8}		Since ef would create a cycle (blue) there is only one available edge at f. Add fg (pink). All other available vertices with deg(v)=1 have at least 2 edges available. We will consider two options available at e: (1) adding eh and (2) adding ce.
{2,3,4,8} Option 1		Add eh (pink). This eliminates edge gh since deg(h)=2. Thus, only one edge is available at g. Add cg (pink). This results in a trivial knot.
{2,3,4,8} Option 2		Add ce (pink). This eliminates edge eh since deg(e)=2. Thus, only one edge is available at h. Add gh (pink). This results in a trivial knot.

{2,3,4,9}		Since ef would create a cycle (blue), there is only one available edge at e. Add ce (pink). This also causes only one edge to be available at f. Add fg (pink). Since deg(g)=2, this eliminates edge cg leaving only one edge available at c. Add cd. This results in a trivial knot.
{2,3,6,n} Series		Note: As described above, the fourth . crossing is 7, 8, or 9, meaning we use edge ei. Then the only possibility at d is adding edge bd. This means we have crossings {2,3,4,6}, a case considered earlier.
{2,3,7,8}		Since ef would create a cycle (blue) there are two available edges at e. All other available vertices with deg(v)=1 have at least 2 edges available. We will consider two options available at e: (1) adding eh and (2) adding ce.

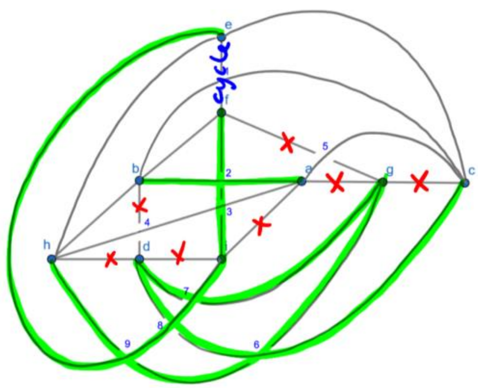
{2,3,7,8} Option 1		Add eh. This eliminates edge ce. Since cg would make a cycle {g,d,c,g}, this leaves only one available edge at c. Add bc (pink). This eliminates edge bf leaving only one available edge at f. Add fg (pink). This results in a trivial knot.
{2,3,7,8} Option 2		Add ce. This eliminates edge bc. Edge fg creates cycle {f,i,e,c,d,g,f} so there is only one available edge af f. Add bf (pink). This also causes there to be only one available edge at g. Add gh (pink). This results in a trivial knot.
{2,3,7,9}		Since ef creates a cycle, there is only one available edge at e. Add ce (pink). There is also only one available edge at f. Add bf (pink). This eliminates bc causing only one edge to be available at c. Add cd (pink). This results in a trivial knot.

{2,3,8,9}		Since ef creates a cycle (blue), there is only one available edge at e. Add edge ce (pink). At b, if we add bd, we have crossings 2,3,4,8, (and 9) a case considered earlier. So add bf (pink). This eliminates fg causing only one edge to be available at g. Add dg (pink). This results in a trivial knot.
Therefore, crossings 2&3 with two additional crossings will not create a non-trivial knot.		
7&8		Crossing 4 is not possible since deg(d)=2 causing edge bd to be unavailable. This leaves 6C2 = 15 possibilities. However {1,2,7,8}, {1,3,7,8} and {2,3,7,8} were considered earlier. This leaves: {1,5,7,8}, {1,6,7,8}, {1,7,8,9}, {2,5,7,8}, {2,6,7,8}, {2,7,8,9}, {3,5,7,8}, {3,6,7,8}, {3,7,8,9}, {5,6,7,8}, {5,7,8,9}, or {6,7,8,9}.

{1,5,7,8}		This combination is not possible since deg(c)=3.
{1,6,7,8}		Since edge fi creates a cycle, there is only one available edge at f. Add bf (pink). This eliminates edges ab and bh leaving only one available edge at h. Add ah (pink). There is only one available edge at a. Add ai (pink). This results in a trivial knot.
{1,7,8,9}		All of the same (green) edges are used for crossing combinations {1,6,7,8} (above) and {1,7,8,9}. Thus, both of these combinations result in trivial knots.

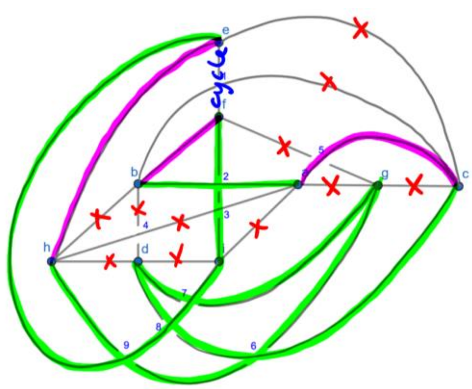
<table>
<tr><td>{2,5,7,8}</td><td></td><td>There is only one edge available at e. Add eh (pink).

There is only one edge available at b since deg(c)=2, deg(d)=2 and deg(f)=2. Add bh (pink).

This results in a trivial knot.</td></tr>
<tr><td>{2,6,7,8}</td><td></td><td>Note that edge ef creates a cycle.

All other vertices with deg(v)=1 have at least two edges to consider.

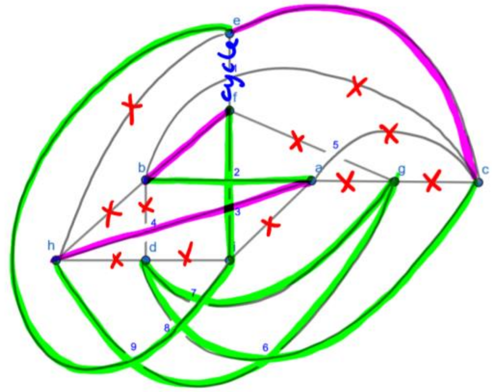
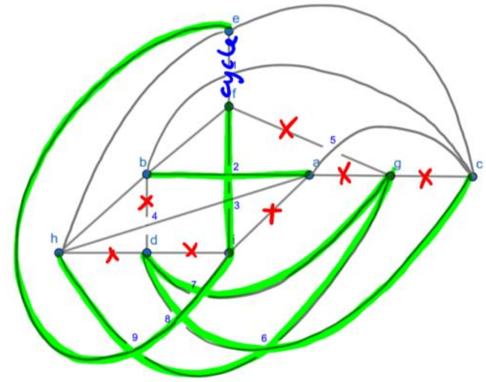
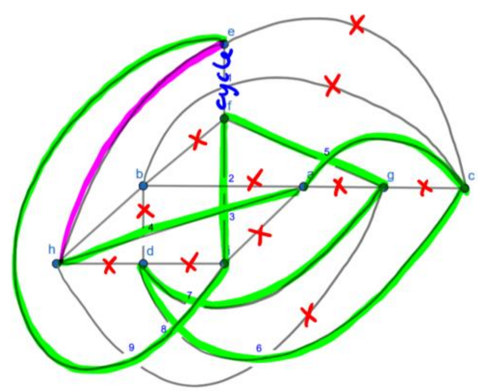
We will consider two options available at e: (1) adding eh and (2) adding ce.</td></tr>
<tr><td>{2,6,7,8}

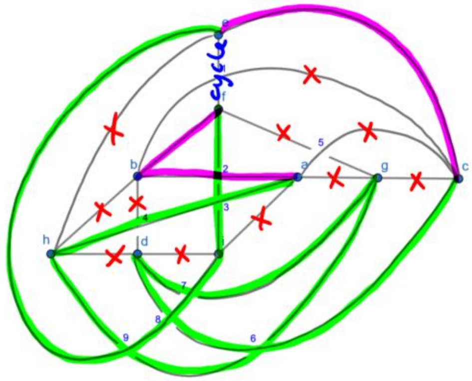
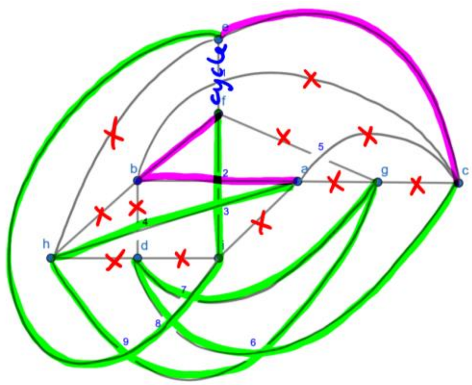
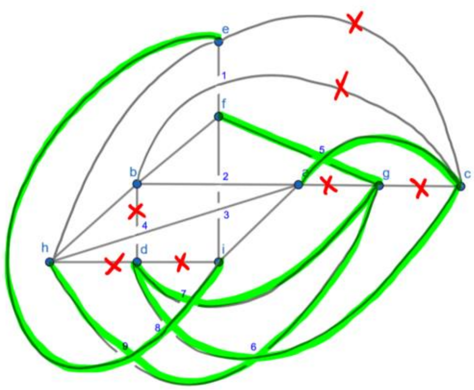
Option 1</td><td></td><td>Add edge eh (pink).

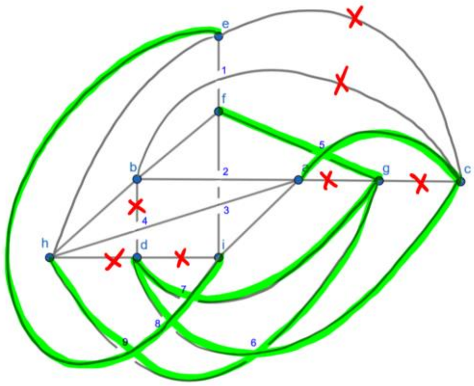
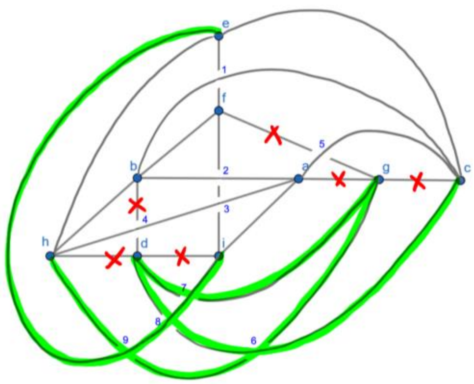
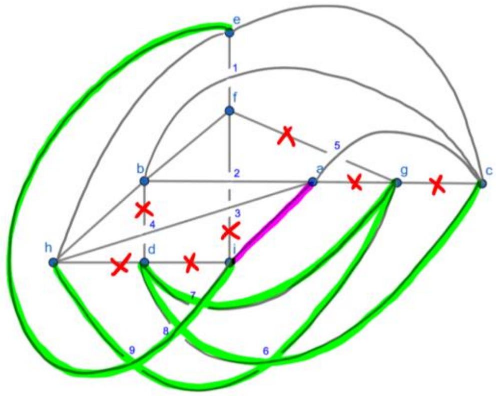
This eliminates edges ah and bh leaving only one edge at a. Add ac (pink).

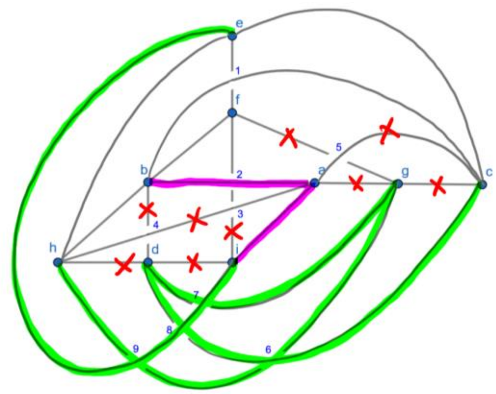
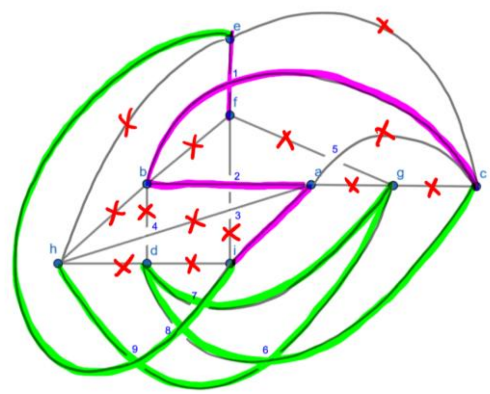
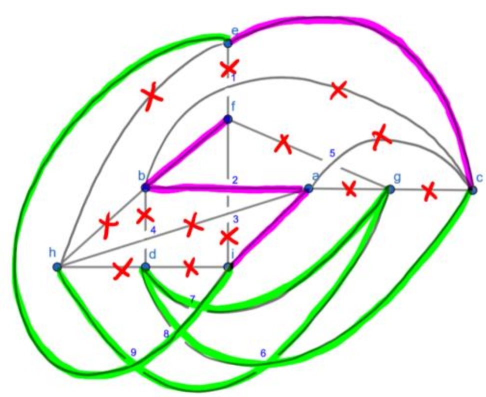
This eliminates edge bc leaving only one available edge at b. Add bf (pink).

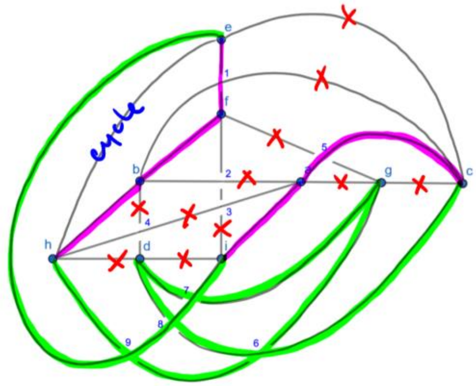
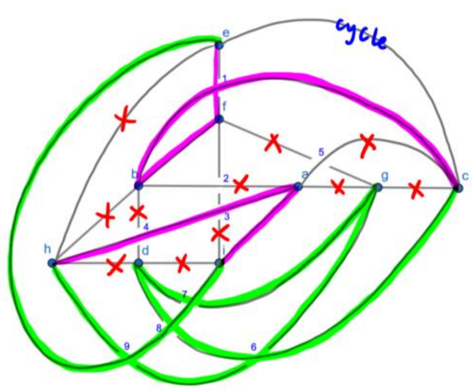
This results in a trivial knot.</td></tr>
</table>

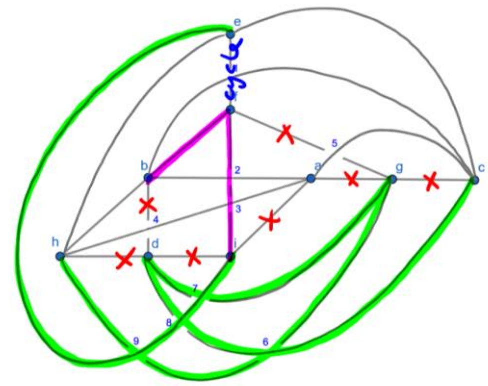
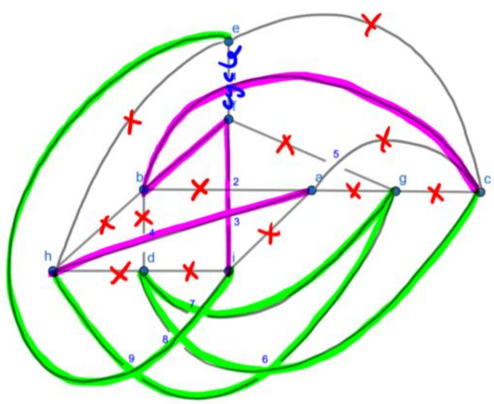
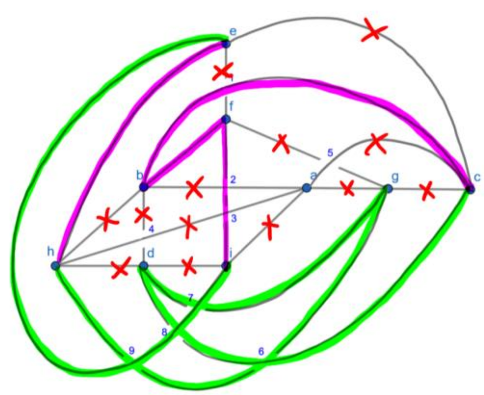
{2,6,7,8} Option 2		Add ce (pink). This eliminates edge ac and bc leaving only one available edge at a. Add ah (pink). This eliminates edges bh and eh leaving only one available edge at b. Add bf (pink). This results in a trivial knot.
{2,7,8,9}		Note that this crossing combination contains all the same (green) edges as {2,6,7,8} above. Thus, both crossing combinations result in trivial knots.
{3,5,7,8}		Since edge ef creates a cycle, there is only one edge available at e. Add eh (pink). This results in a trivial knot.

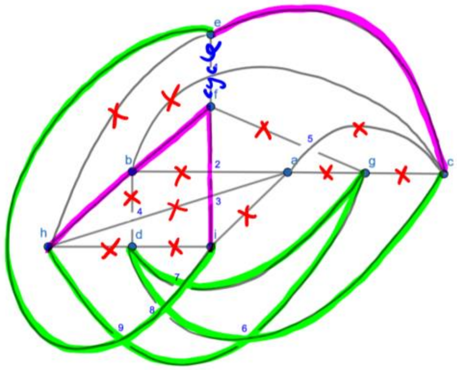
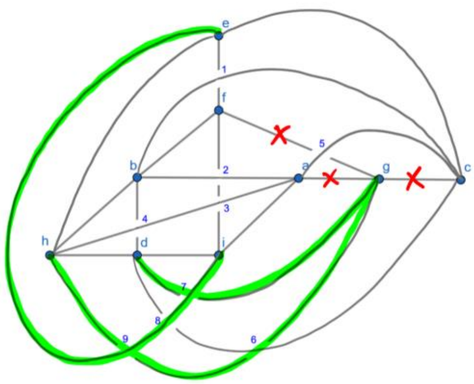
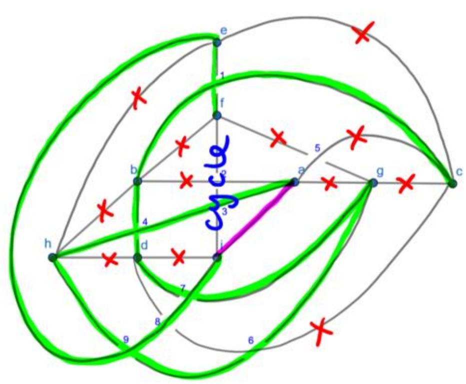
{3,6,7,8}		Since edge ef creates a cycle, there is only one available edge at e. Add ce (pink). This eliminates edges ac and bc. There is also only one edge available at f. Add bf (pink). This leaves only one available edge at b. Add ab (pink). This results in a trivial knot.
{3,7,8,9}		Note that this crossing combination contains all the same (green) edges as {3,6,7,8} above. Thus, both crossing combinations result in trivial knots.
{5,6,7,8}		This crossing combination is not possible as deg(g)=3.

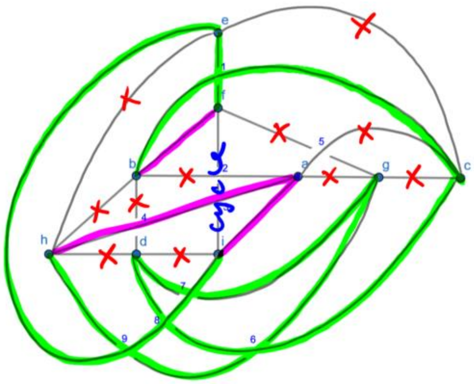
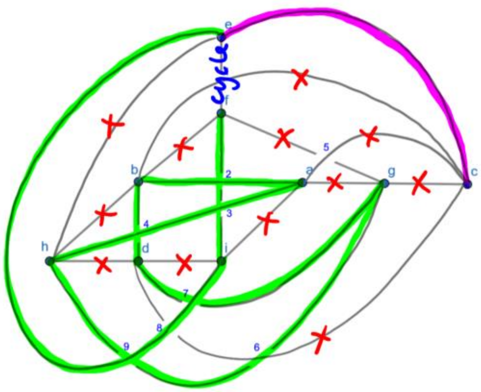
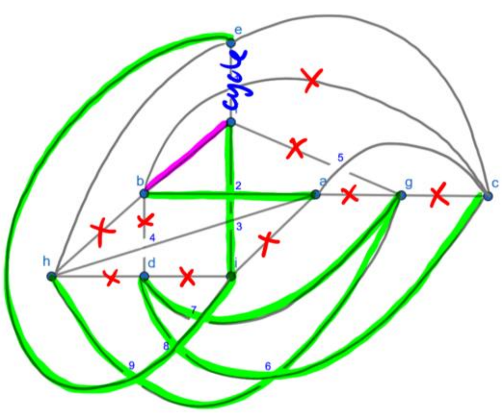
{5,7,8,9}		This crossing combination is not possible as deg(g)=3.
{6,7,8,9}		Note that all vertices with deg(v)=1 have at least two edges to consider. We will consider two options available at i: (1) adding ai and (2) adding fi.
{6,7,8,9} Option 1		Add ai (pink). This still leaves all vertices with deg(v)=1 have at least two edges to consider. We will consider three options available at a: (1a) adding ab, (1b) adding ac and (1c) adding ah.

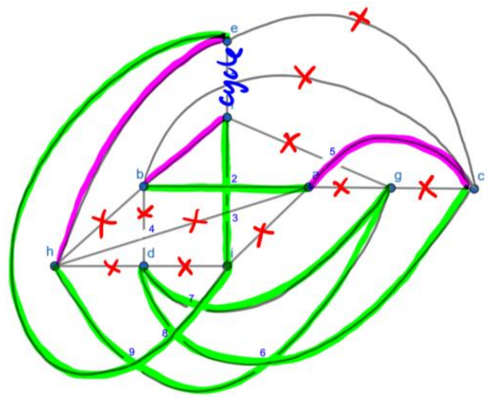
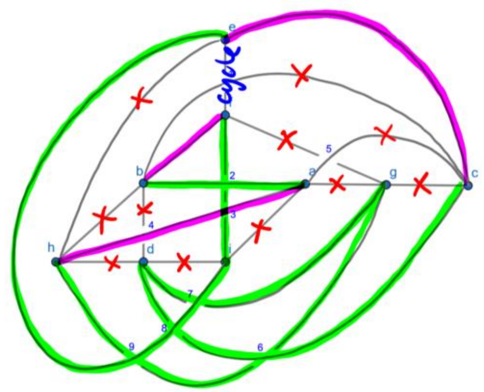
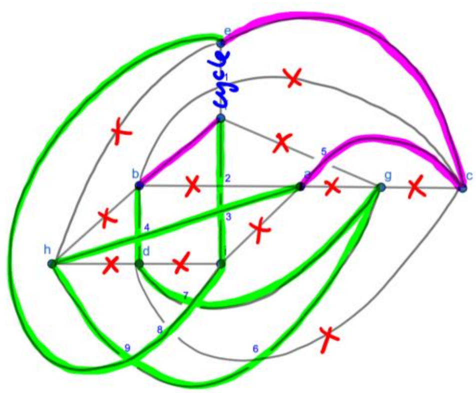
{6,7,8,9} Option 1a		Add ab (pink). This still leaves all vertices with deg(v)=1 have at least two edges to consider. We will consider two options available at c: (1ai) adding bc and (1aii) adding ce.
{6,7,8,9} Option 1ai		Add bc (pink). This eliminates edge bf leaving only one available edge at f. Add ef (pink). This eliminates edge eh leaving deg(h)=1. Thus, this does not result in a knot.
{6,7,8,9} Option 1aii		Add ce (pink). This eliminates edges ef and eh leaving only one edge available at b. Add bf (pink). This eliminates edge bh causing deg(h)=1. Thus, this does not result in a knot.

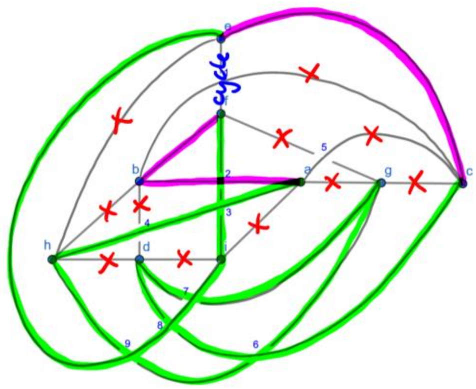
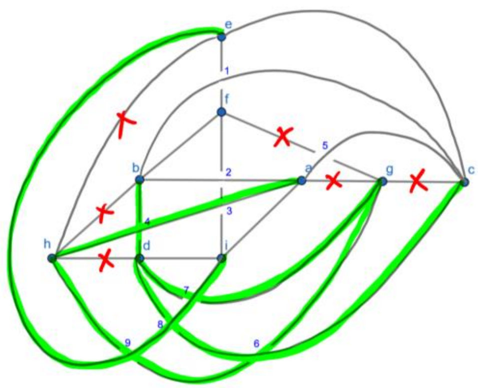
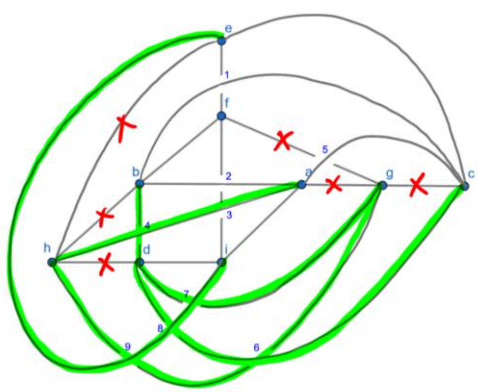
{6,7,8,9} Option 1b		Add ac (pink). This eliminates edges bc, ce, and cg. The edge eh creates cycle {eiacdghe} leaving only one edge available at e. Add ef (pink). There is only one edge available at f. Add bf (pink). There is only one edge available at h. Add bh (pink). This results in a trivial knot.
{6,7,8,9} Option 1c		Add ah (pink). This eliminates edge bh and eh. Since edge ce creates cycle {eiahgdce} there is only one edge available at e. Add ef (pink). There is only one edge available at f. Add bf (pink). There is only one available edge at b. Add bc (pink). This results in a trivial knot.

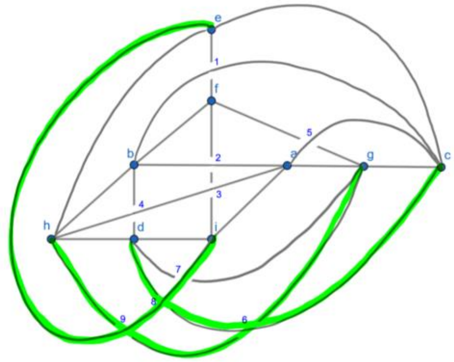
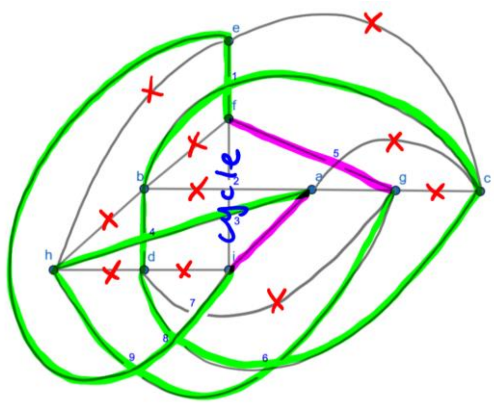
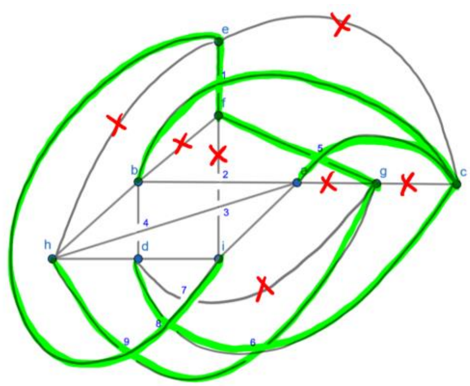
{6,7,8,9} Option 2		Add fi (pink). Since edge ef creates a cycle there is only one edge available at f. Add bf (pink). This still leaves all vertices with deg(v)=1 have at least two edges to consider. We will consider three options available at b: (2a) adding ab (2b) adding bc and (2c) adding bh.
{6,7,8,9} Option 2a		Add ab (pink). This eliminates edges ab, bh and eh leaving b with only one available edge. Add bc (pink). This eliminates edges ac and ce causing deg(a)=deg(e)=1. This does not result in a knot.
{6,7,8,9} Option 2b		Add bc (pink). This eliminates edge ac and ce leaving only one available edge at e. Add eh (pink). This results in a trivial knot.

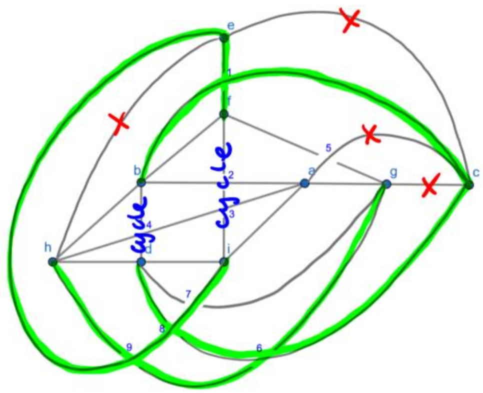
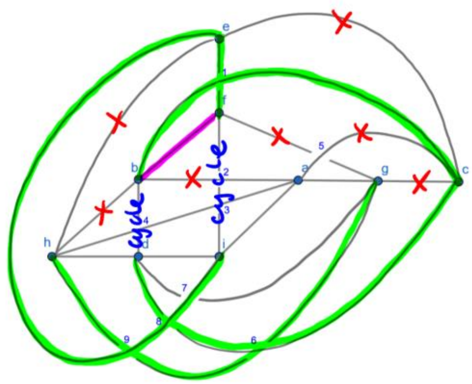
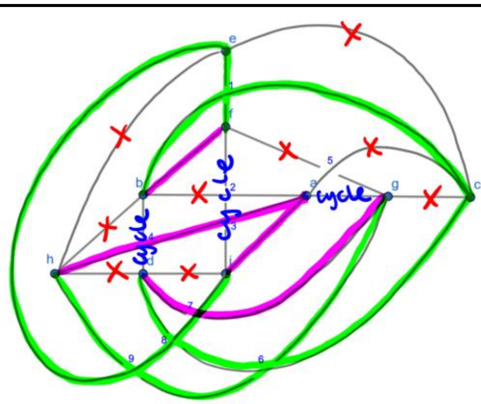
{6,7,8,9} Option 2c		Add bh (pink). This eliminates edges ah and eh leaving only one available edge at e. Add ce (pink). This results in a trivial knot.
Therefore, crossings 7&8 with two additional crossings will not create a non-trivial knot.		
7&9		Crossing 5 is not possible since deg(g)=2 causing edge fg to be unavailable. This leaves 6C2 = 15 possibilities. However {1,2,7,9}, {1,3,7,9}, {1,7,8,9}, {2,3,7,9}, {2,7,8,9}, {3,7,8,9}, {6,7,8,9} were considered earlier. This leaves: {1,4,7,9}, {1,6,7,9}, {2,4,7,9}, {2,6,7,9}, {3,4,7,9}, {3,6,7,9}, {4,6,7,9}, or {4,7,8,9}.
{1,4,7,9}		Since edge fi creates a cycle, there is only one edge available at i. Add ai (pink). This eliminates edge ac leaving deg(c)=1. This does not result in a knot.

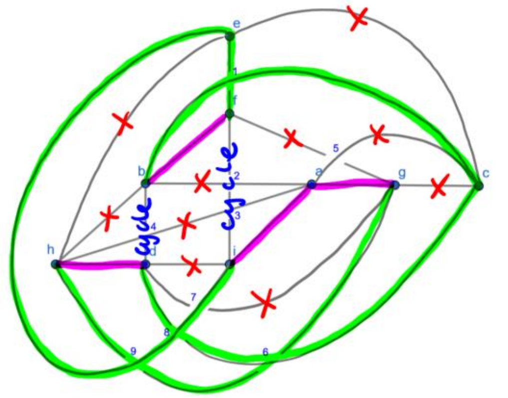
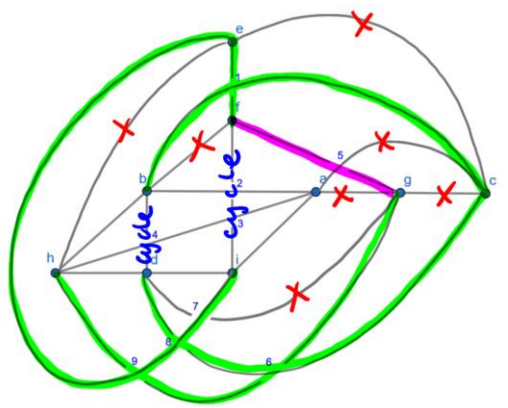
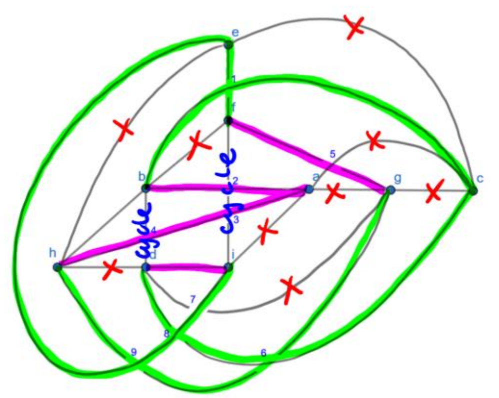
{1,6,7,9}		Since edge fi creates a cycle, there is only one edge available at i. Add ai (pink). Also, there is only one edge available at f. Add bf (pink). This leaves only one available edge at a. Add ah (pink). This results in a trivial knot.
{2,4,7,9}		Since ef creates a cycle, there is only one available edge at e. Add ce (pink). There are no other edges available at c so deg(c)=1. This does not result in a knot.
{2,6,7,9}		Since edge ef creates a cycle there is only one edge available at f. Add bf (pink). All other vertices have at least two edges available. We will consider two options available at e: (1) adding eh and (2) adding ce.

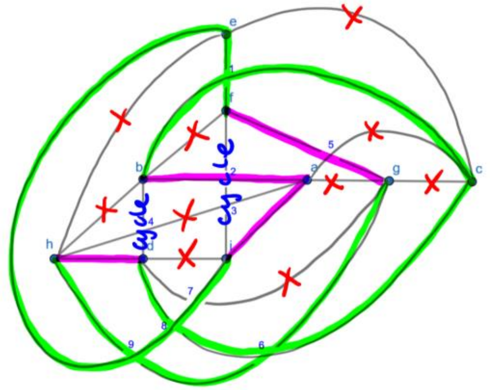
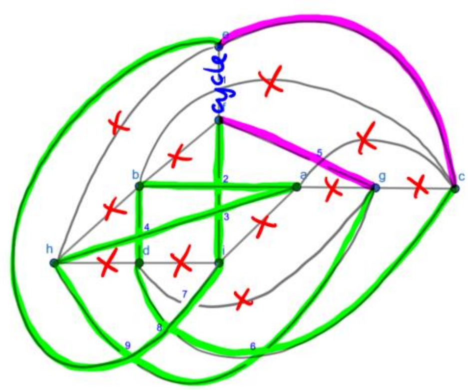
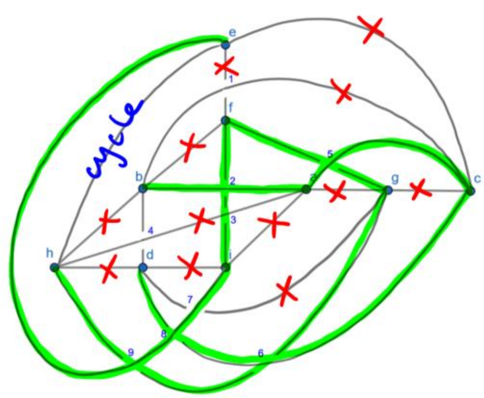
{2,6,7,9} Option 1		Add edge eh (pink). This eliminates edge ah leaving only one available edge at a. Add ac (pink). This results in a trivial knot.
{2,6,7,9} Option 2		Add ce (pink). This eliminates edge eh leaving only one available edge at h. Add ah (pink). This results in a trivial knot.
{3,4,7,9}		Since ef creates a cycle there is only one available edge at e. Add ce (pink). Also, there is only one available edge at f. Add bf (pink). This eliminates edges ab and bc leaving only one available edge at c. Add ac (pink). This results in a trivial knot.

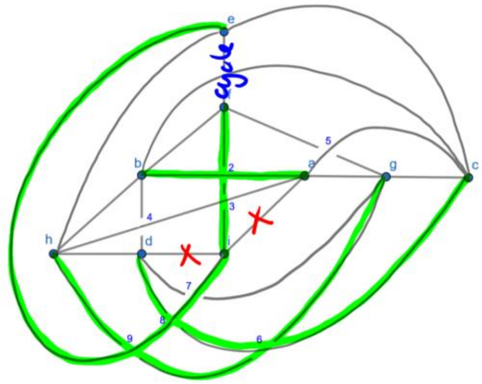
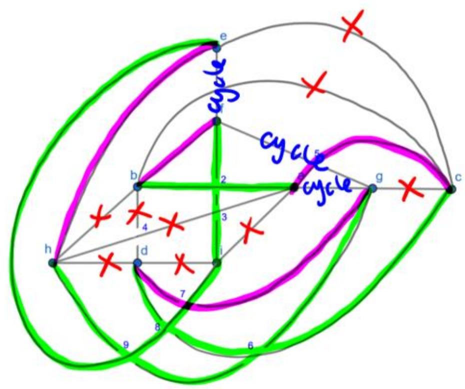
{3,6,7,9}		Since ef creates a cycle there is only one available edge at e. Add ce (pink). This eliminates edges bc. There is only one available edge at f. Add bf (pink). This leaves only one available edge at b. Add ab (pink). This results in a trivial knot.
{4,6,7,9}		This crossing results in deg(d)=3. Thus, this does not result in a knot.
{4,7,8,9}		This crossing combination includes the same (green) edges as {4,6,7,9}. Neither combination results in a knot.
Therefore, crossings 7&9 with two additional crossings will not create a non-trivial knot.		

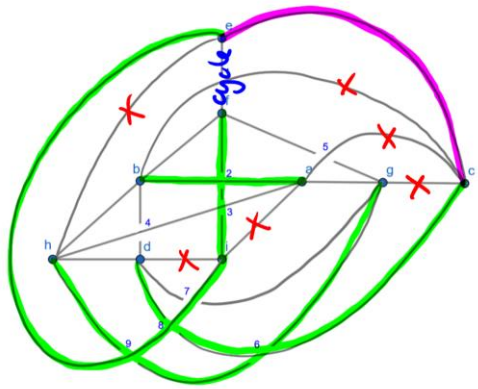
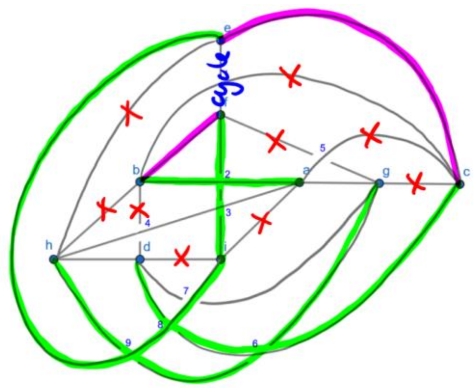
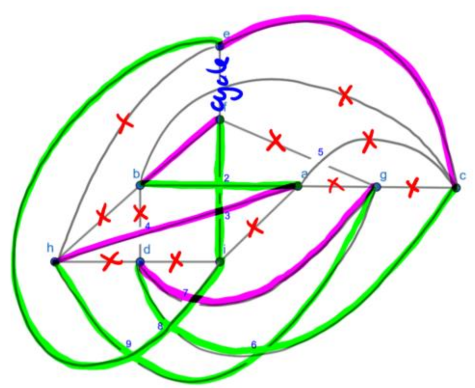
8&9		All 7 other crossings are possible since none of the other edges are eliminated for deg(v)=2. This leaves 7C2 = 21 possibilities. However {1,2,8,9}, {1,3,8,9}, {1,7,8,9} , {2,3,8,9}, {2,7,8,9}, {3,7,8,9}, {4,7,8,9}, {5,7,8,9} and {6,7,8,9} were considered earlier. This leaves: {1,4,8,9}, {1,5,8,9}, {1,6,8,9}, {2,4,8,9}, {2,5,8,9}, {2,6,8,9}, {3,4,8,9}, {3,5,8,9}, {3,6,8,9}, {4,5,8,9}, {4,6,8,9}, or {5,6,8,9}.
{1,4,8,9}		Since edge fi creates a cycle there is only one edge available at f. Add fg (pink). This also allows for only one available edge at i. Add ai (pink). This results in a trivial knot.
{1,5,8,9}		This combination of crossings causes deg(c)=3. Thus, this will not result in a knot.

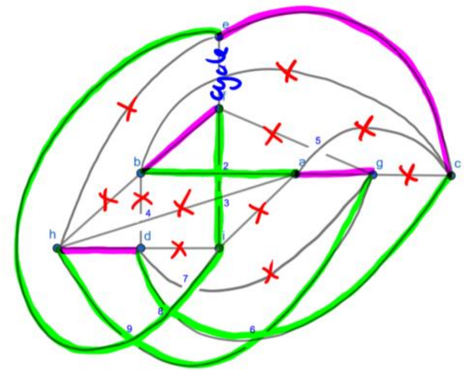
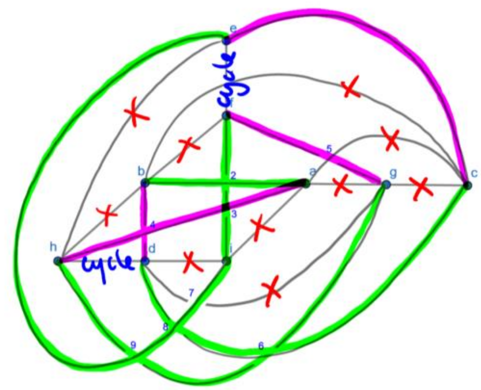
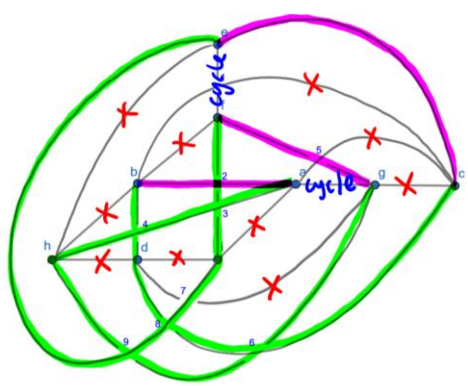
{1,6,8,9}		Since edge fi creates cycle {eife} edge fi is unavailable. Similarly, edge bd creates cycle {bcdb} so edge bd is unavailable. All other vertices have at least two edges available. We will consider two options available at f: (1) adding bf and (2) adding fg.
{1,6,8,9} Option 1		Add bf (pink). This eliminates edges ab and bh. All other vertices have at least two edges available. We will consider two options available at h: (1a) adding ah and (1b) adding dh.
{1,6,8,9} Option 1a		Add ah (pink). This eliminates edge dh. Edge ag creates cycle {ahga} so ag is unavailable leaving only one available edge at g. Add dg (pink). This eliminates edge di leaving only one available edge at i. Add ai (pink). This results in a trivial knot.

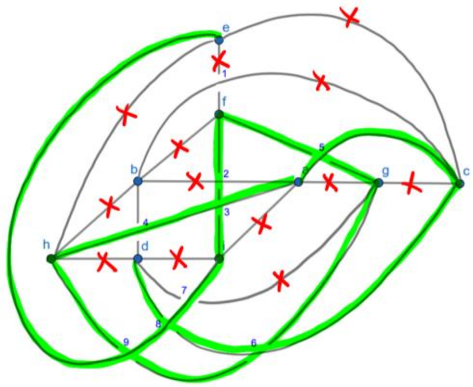
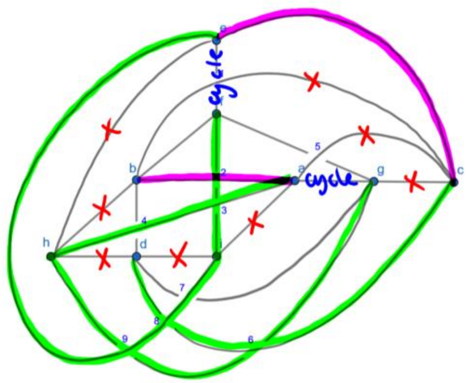
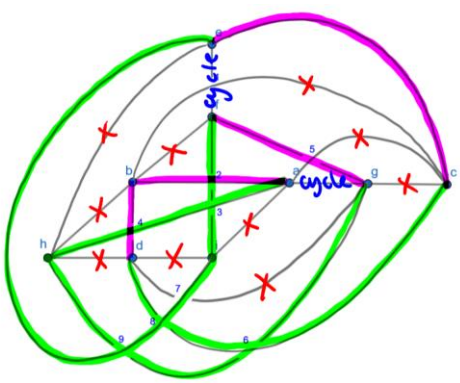
{1,6,8,9} Option 1b		Add dh (pink). This eliminates edge ah, dg and di. There is only one edge available at g. Add ag (pink). There is only one available edge at a. Add ai (pink). This results in a trivial knot.
{1,6,8,9} Option 2		Add fg (pink). This eliminates edges bf, ag and dg. All other vertices have at least two edges available. We will consider two options available at i: (2a) adding di and (2b) adding ai.
{1,6,8,9} Option 2a		Add di (pink). There are two ways to connect the remaining degree 1 vertices b and h. Either via vertex a (shown - add ab and ah in pink) or else add the edge bh. Both choices result in trivial knots.

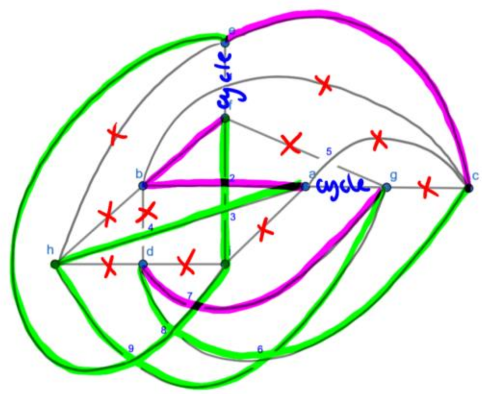
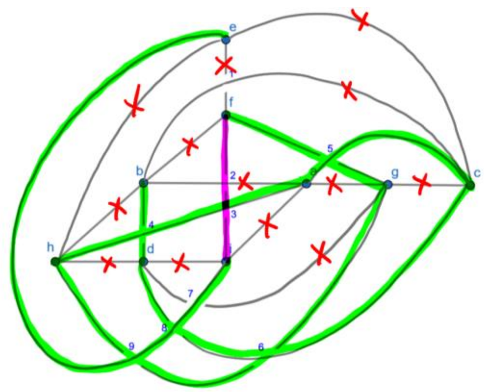
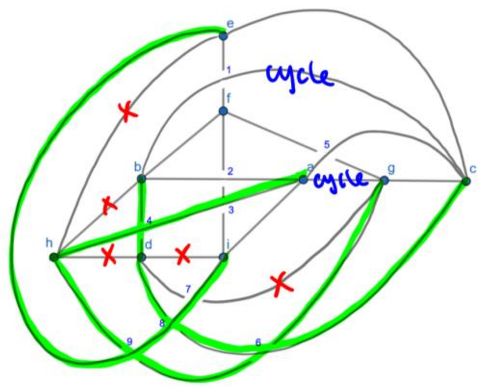
{1,6,8,9} Option 2b		Add ai (pink). This eliminates edges di leaving only one edge available at d. Add dh (pink). This eliminates edge ah leaving only one edge available at a. Add ab (pink). This results in a trivial knot.
{2,4,8,9}		Since edge ef creates a cycle there is only one edge available at e. Add ce (pink). This eliminates edge cg leaving only one available edge at g. Add fg (pink). This results in a trivial knot.
{2,5,8,9}		Since edge eh creates cycle {eifghe} edge eh is unavailable. This causes deg(e)=1. Thus, this does not result in a knot.

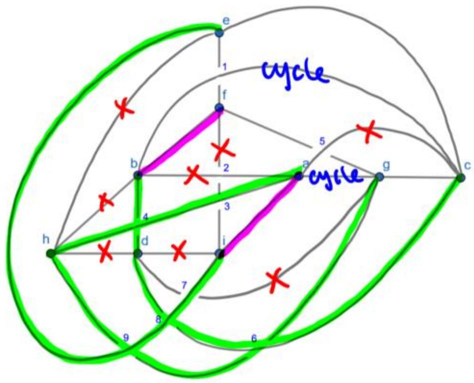
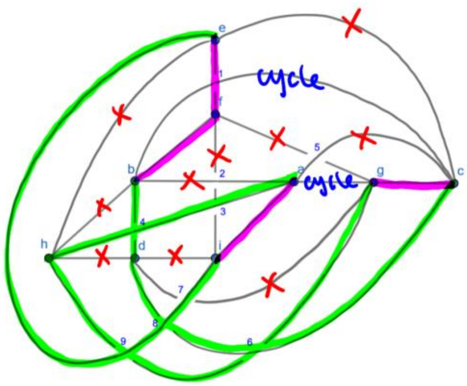
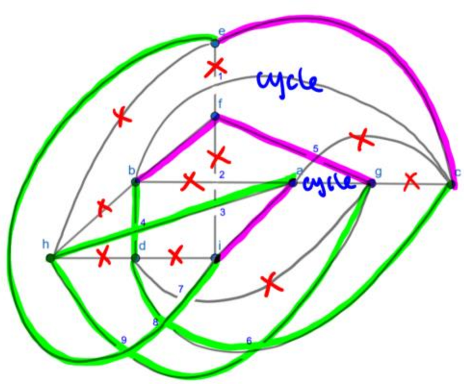
{2,6,8,9}		Since edge ef creates a cycle it is unavailable. All other vertices have at least two edges available. We will consider two options available at e: (1) adding eh and (2) adding ce.
{2,6,8,9} Option 1		Add eh (pink). Since edge fg creates cycle {fiehgf} the edge fg is unavailable. There is only one edge at f. Add bf (pink). Since edge ag creates cycle {acdga} the edge ag is unavailable. There is only one edge available at a. Add ac (pink). This eliminates cg. There is only one edge at g. Add dg (pink). This is a trivial knot.

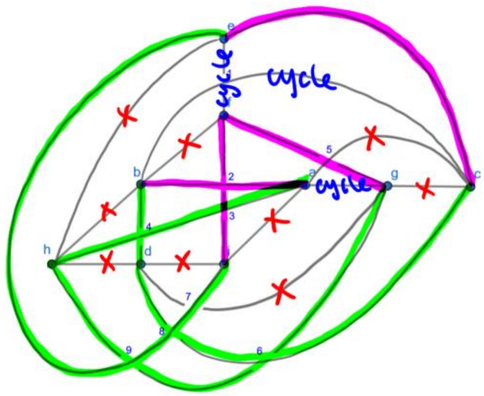
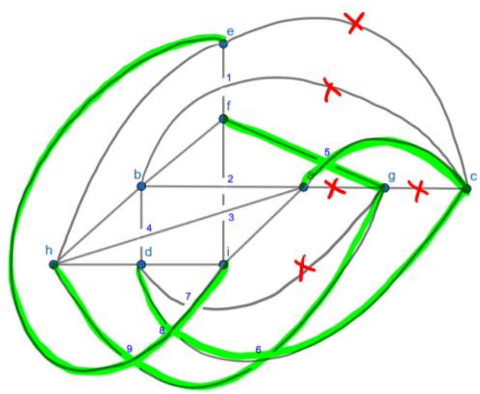
{2,6,8,9} Option 2		Add ce (pink). This eliminates edge eh. All other vertices have at least two edges available. We will consider two options available at f: (2a) adding bf and (2b) adding fg.
{2,6,8,9} Option 2a		Add bf (pink). This eliminates edges bd, bh, and fg. All other vertices have at least two edges available. We will consider two options available at h: (2ai) adding ah and (2aii) adding dh.
{2,6,8,9} Option 2ai		Add ah (pink). This eliminates dh leaving only one available edge at d. Add dg (pink). This results in a trivial knot.

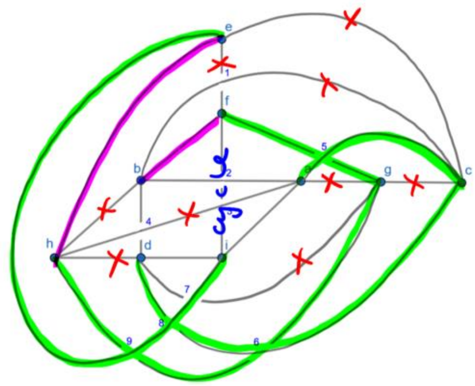
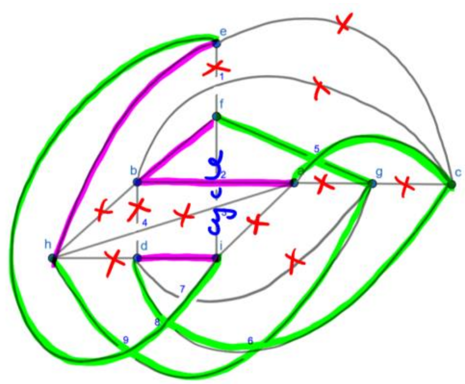
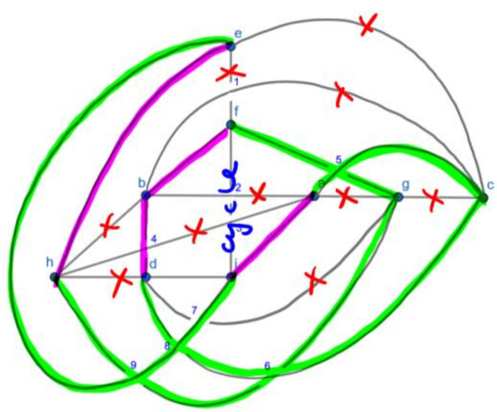
{2,6,8,9} Option 2aii		Add dh (pink). This eliminates dg leaving only one edge available at g. Add ag (pink). This results in a trivial knot.
{2,6,8,9} Option 2b		Add fg (pink). Edge dh creates cycle {hgfiecdh} so dh is unavailable leaving only one edge at d. Add bd (pink). This eliminates edge bh leaving only one available edge at h. Add ah (pink). This results in a trivial knot.
{3,4,8,9}		Edge ef creates a cycle so only one edge is available at e. Add ce (pink). This eliminates bc, ac, and cg. Edge ag creates cycle {ahga} so ag is unavailable leaving only one edge at g. Add fg (pink). This eliminates bf leaving only one edge available at b. Add ab (pink). This results in a trivial knot.

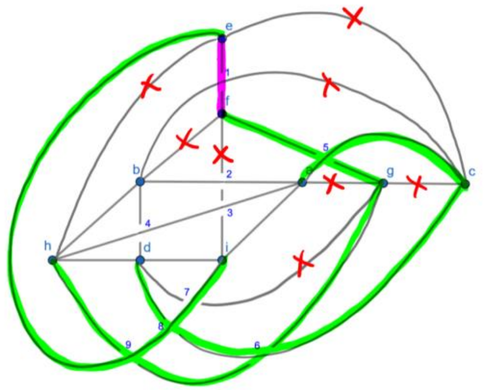
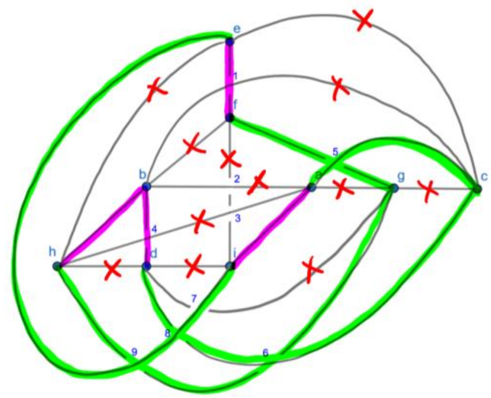
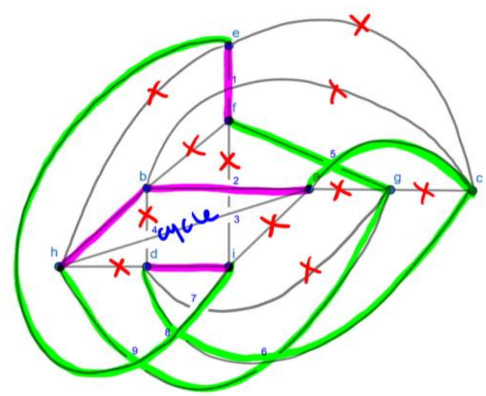
{3,5,8,9}		This combination of crossings results in deg(e)=1 since deg(c)=deg(f)=deg(h)=2. Thus, this does not result in a knot.
{3,6,8,9}		Since ef creates a cycle, there is only one edge available at e. Add ce (pink). Edge ag creates cycle {ahga} so ag is also unavailable. All other vertices have at least two edges available. We will consider two options available at b: (1) adding bd and (2) adding bf.
{3,6,8,9} Option 1		Add bd (pink). This eliminates edge dg leaving only one edge available at g. Agg fg (pink). This results in a trivial knot.

{3,6,8,9} Option 2		Add bf (pink). This eliminates edge bd leaving only one edge available at d. Add dg (pink). This results in a trivial knot.
{4,5,8,9}		Since there is only one edge available at i, add fi (pink). This eliminates ef causing deg(e)=1. Thus, this does not result in a knot.
{4,6,8,9}		Edge bc creates cycle {bcdb} and edge ag creates cycle {ahga} so both of these edges are unavailable. All other vertices have at least two edges available. We will consider two options available at i: (1) adding ai and (2) adding fi.

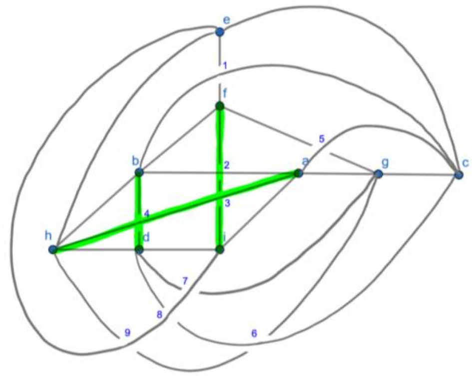
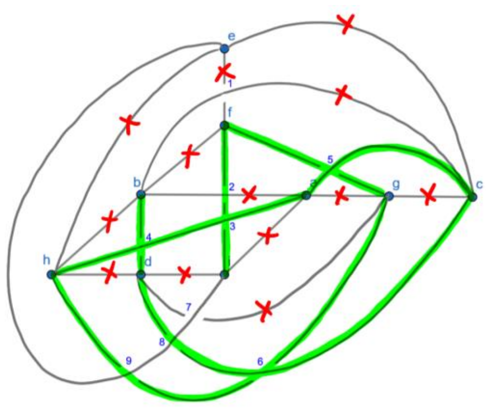
{4,6,8,9} Option 1		Add ai (pink). This eliminates edge ab and ac. There is only one edge available at b. Add bf (pink). All other vertices have at least two edges available. We will consider two options available at f: (1a) adding ef and (1b) adding fg.
{4,6,8,9} Option 1a		Add ef (pink). This eliminates edge ce leaving only one edge available at c. Add cg (pink). This results in a trivial knot.
{4,6,8,9} Option 1b		Add fg (pink). This eliminates edge ef leaving only one available edge at e. Add ce (pink). This results in a trivial knot.

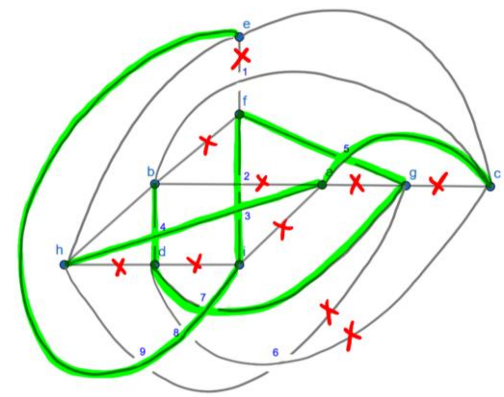
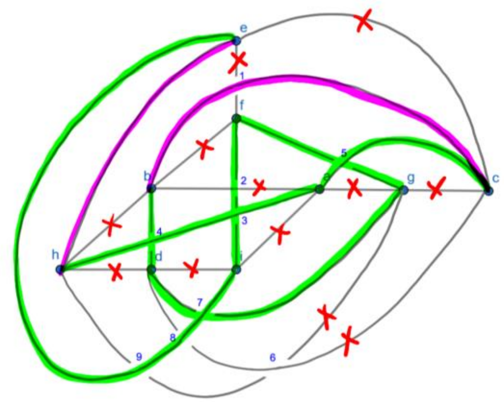
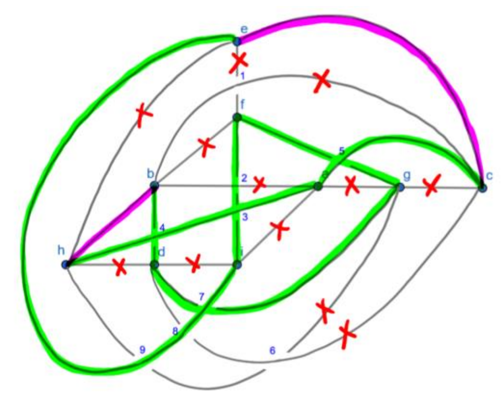
{4,6,8,9} Option 2		Add fi (pink). This eliminates edge ai. Edge ef would create a cycle so there is only one available edge at e. Add ce (pink). This eliminates edges ac and cg. There is only one available edge at g. Add fg (pink). Similarly, there is only one edge available at a. Add ab (pink). This results in a trivial knot.
{5,6,8,9}		Aside from c and g, all other vertices have at least two edges available. We will consider two options available at e: (1) adding eh and (2) adding ef.

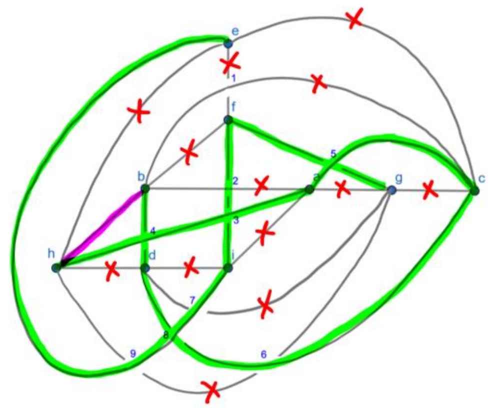
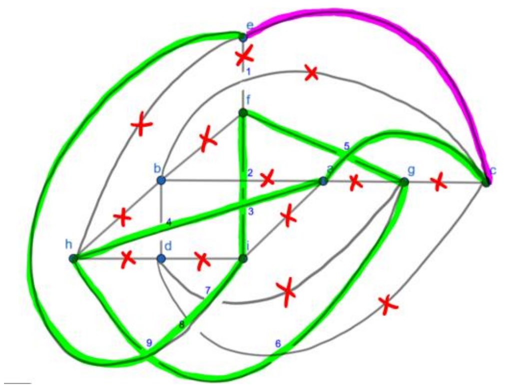
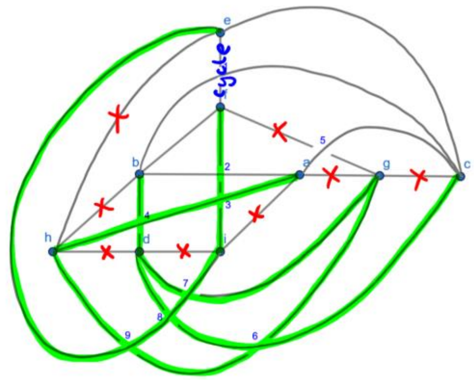
{5,6,8,9} Option 1		Add eh (pink). This eliminated edge ef. Edge fi creates cycle {fgheif} so fi is unavailable leaving only one edge at f. Add bf (pink). All other vertices have at least two edges available. We will consider two options available at a: (1a) adding ab and (1b) adding ai.
{5,6,8,9} Option 1a		Add ab (pink). This eliminates edge ai leaving only one edge available at i. Add di (pink). This results in a trivial knot.
{5,6,8,9} Option 1b		Add ai (pink). This eliminates edge ab leaving only one edge available at b. Add bd (pink). This results in a trivial knot.

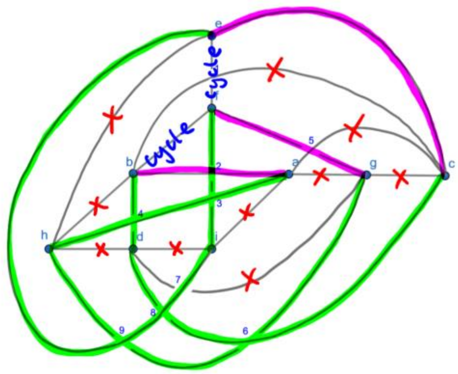
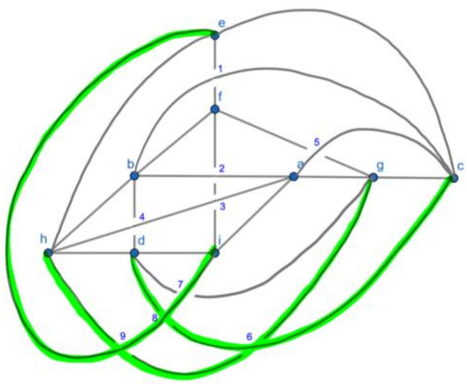
{5,6,8,9} Option 2		Add ef (pink). All other vertices have at least two edges available. We will consider two options available at i: (2a) adding ai and (2b) adding di.
{5,6,8,9} Option 2a		Add ai (pink). This eliminates edges ab, ah and di. This leaves only one available edge at h. Add bh (pink). There is only one available edge at b. Add bd (pink). This results in a trivial knot.
{5,6,8,9} Option 2b		Add di (pink). This eliminates edges ai, bd and dh. This leaves only one available edge at h. Add bh (pink). There is only one available edge at b. Add ab (pink). This results in a trivial knot.

Therefore, crossings 8&9 with two additional crossings will not create a non-trivial knot.

3&4		No additional edges will be eliminated since each vertex included in these crossings have deg(v)=1. This leaves 7C2 = 21 possibilities. However, crossings {1,2,3,4}, {1,3,4,5}, {1,3,4,6}, {1,3,4,7}, {1,3,4,8}, {1,3,4,9}, {2,3,4,5}, {2,3,4,6}, {2,3,4,7}, {2,3,4,8}, {2,3,4,9}, {3,4,7,9} and {3,4,8,9} were covered above. Note {3,4,7,8} is not possible since this would cause deg(d)=3. This leaves: {3,4,5,6}, {3,4,5,7}, {3,4,5,8}, {3,4,5,9}, {3,4,6,7}, {3,4,6,8} and {3,4,6,9}.
{3,4,5,6}		This crossing causes deg(b)=1. Thus, this will not result in a knot.

{3,4,5,7}		Aside from a, d, f, g and i, all other vertices have at least two edges available. We will consider two options available at e: (1) adding eh and (2) adding ce.
{3,4,5,7} Option 1		Add eh (pink). This eliminates edge bh and ce leaving only one available edge at b. Add bc (pink). This results in a trivial knot.
{3,4,5,7} Option 2		Add ce (pink). This eliminates edge eh leaving only one available edge at h. Add bh (pink). This results in a trivial knot.

{3,4,5,8}		Since deg(a)=deg(c)=deg(f)=2, there is only one available edge at b. Add bh (pink). This results in a trivial knot.
{3,4,5,9}		Since deg(f)=deg(h)=2, there is only one available edge at e. Add ce (pink). This results in a trivial knot.
{3,4,6,7}		This combination of crossings causes deg(d)=3. Thus, they will not result in a knot.

{3,4,6,8}		Note that crossings {3,4,6,8} and {3,4,6,9} share the same (green) edges. Since ef creates a cycle, there is only one available edge at e. Add ce (pink). This eliminates ac, bc, and cg. The edge bf creates cycle {bdceifb} leaving only one edge available at f. Add fg (pink). This eliminates edge ag leaving only one available edge at a. Add ab (pink). This results in a trivial knot for both {3,4,6,8} and {3,4,6,9}.
Therefore, crossings 3&4 with two additional crossings will not create a non-trivial knot.		
6&8		As shown in figure, a cycle that includes crossingss 6 & 8 will also include crossing 9. Such cycles were considered above when we examined cycles with crossings 8 & 9.
Therefore, crossings 6&8 with two additional crossings will not create a non-trivial knot.		

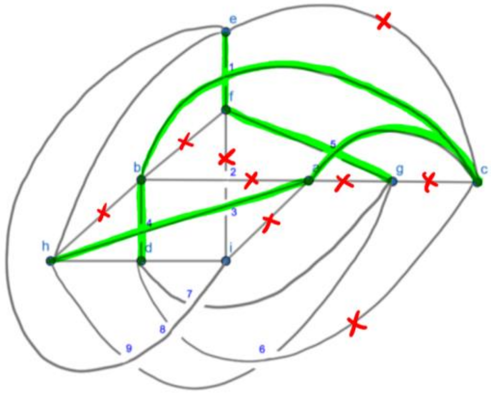
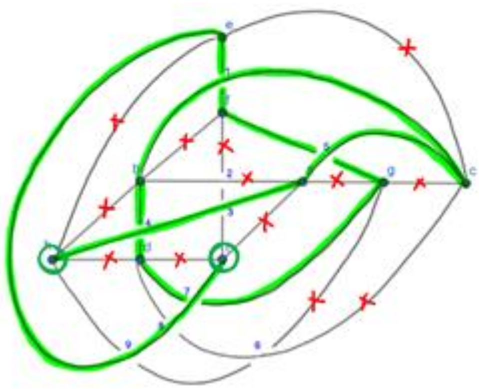
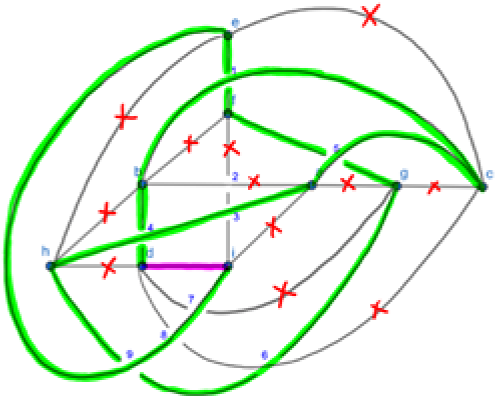
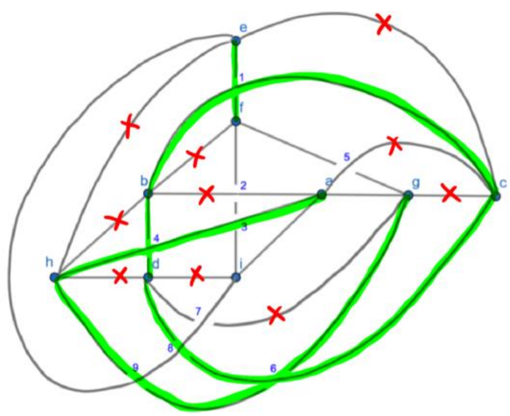
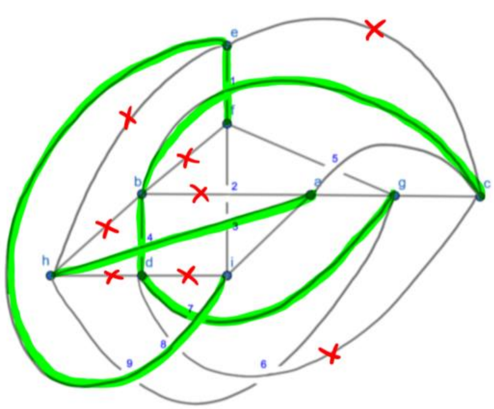
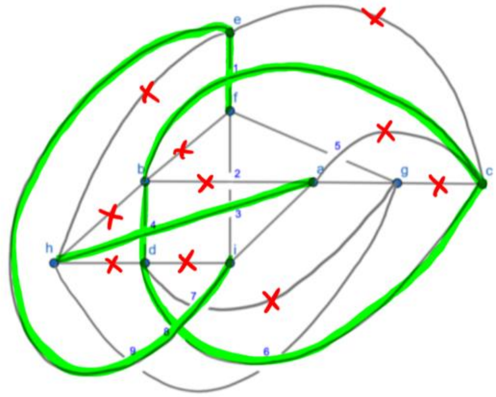
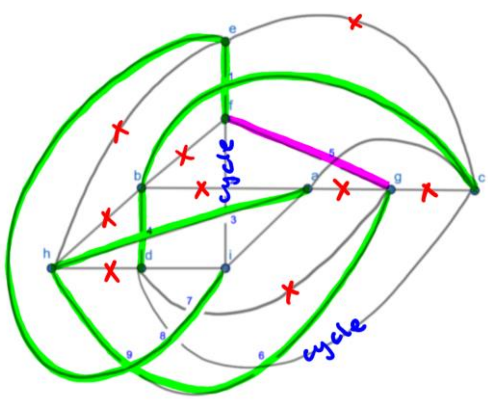
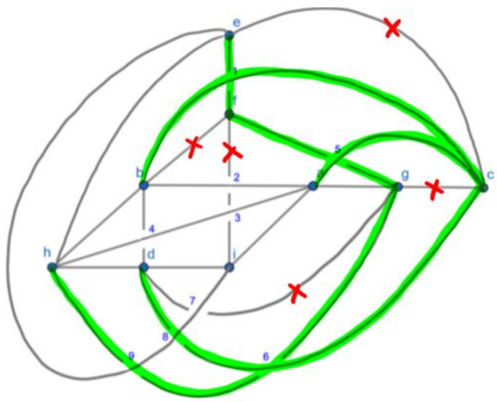
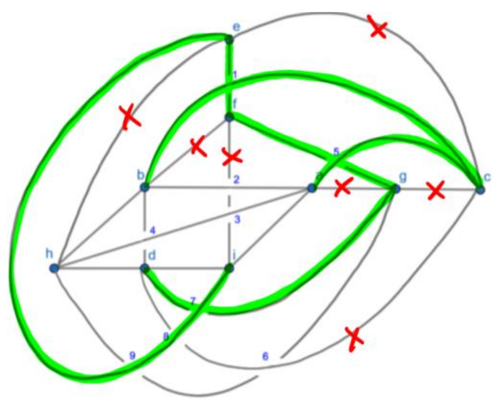
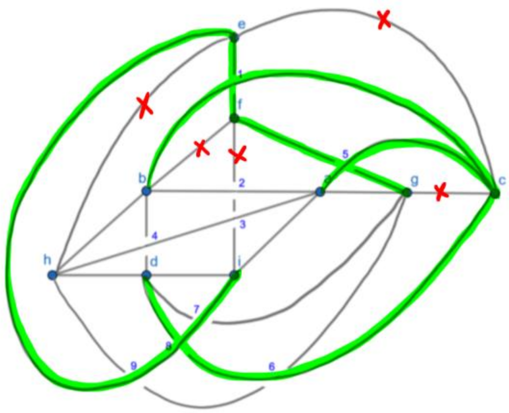
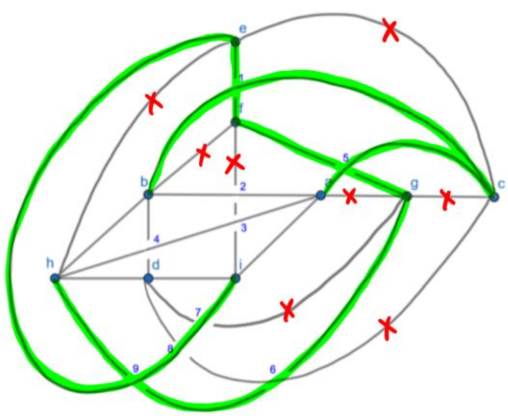
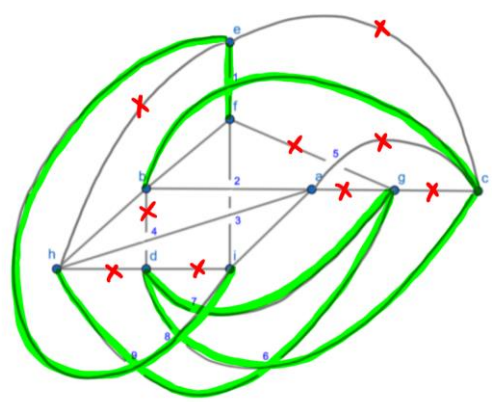
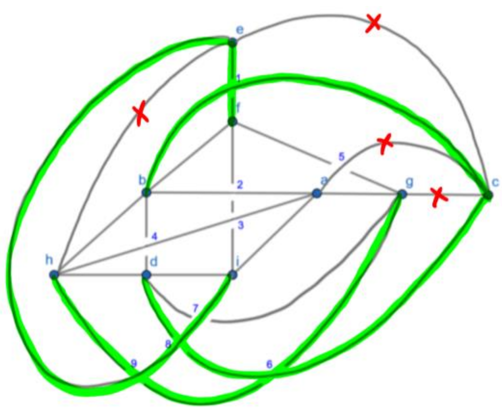
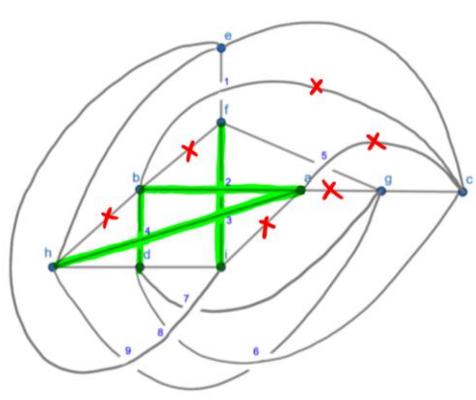
Triple	Diagram	Explanation
{1,4,5}		Since crossings 1 and 5 are consecutive non-alternating crossings at least one other crossing is required for a non-trivial knot. {1,**2**,4,5}: Contains non-alternating crossings 1&2, covered in Appendix A. {1,**3**,4,5}: Contains non-alternating crossings 1&3, covered Appendix A. {1,4,5,**6**}: Edge cd is unavailable since deg(c)=2. {1,4,5,**7**}: See *Figure D1* below. {1,4,5,8}: Edge cd is unavailable since deg(c)=2. {1,4,5,**9**}: See *Figure D2* below.
Figure D1		{1,4,5,7}: After adding edges dg and ei to create crossing 7, no other edges are available to create a knot since deg(d)=deg(e)=deg(g)=2 leaving deg(h)=1 and deg(i)=1. There is no knot with these four crossings.

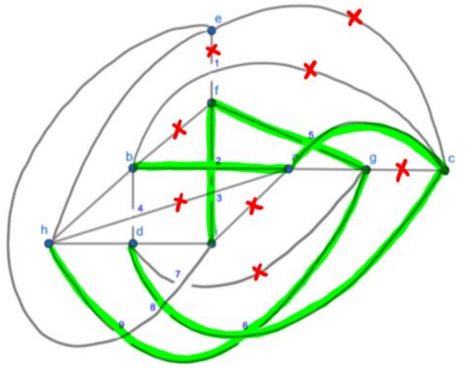
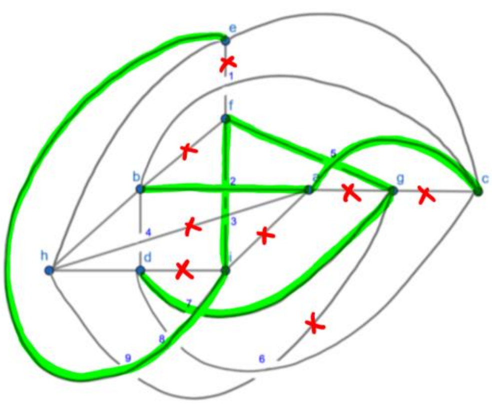
Figure D2		{1,4,5,9}: After adding edges gh and ei to create crossing 9, only one edge is available to connect vertices d and i (highlighted in pink). This creates a cycle in the graph which, through the use of Reidemeister moves, is revealed to be a trivial knot.
Cycles with crossings {1,4,5} will not result in a non-trivial knot.		
{1,4,6}		This set of crossings creates cycle {bcdb} so it is not possible to create a knot with {1,4,6}.
Cycles with crossings {1,4,6} will not result in a non-trivial knot.		
{1,4,7}		Since crossings 4 and 7 are consecutive non-alternating crossings at least one other crossing is required for a non-trivial knot. {1,**2**,4,7}, {1,**3**,4,7}, {1,4,7,**8**} and {1,4,7,**9**} have two consecutive non-alternating crossings already covered in Appendix A. {1,4,**5**,7} and {1,4,**6**,7} were covered above.

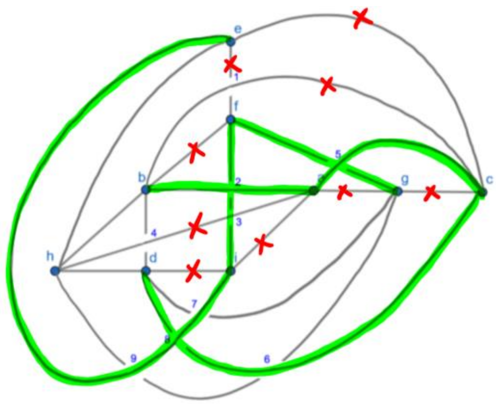
Cycles with crossings {1,4,7} will not result in a non-trivial knot.		
{1,4,8}		This set of crossings creates cycle {bcdb} so it is not possible to create a knot with {1,4,8}.
Cycles with crossings {1,4,8} will not result in a non-trivial knot.		
{1,4,9}		At f, we cannot add edge fi as that will give a cycle. Add edge fg (pink). This eliminates edges ag, cg and dg. At c, we cannot add edge cd as that will give a cycle. This leaves only one edge available at c. Adding edge ac would create crossing 5. Cycles with crossings {1,4,5,9} are covered above.
Cycles with crossings {1,4,9} will not result in a non-trivial knot.		

{1,5,6}		It is not possibly to have a knot with this crossing combination since deg(c)=3.
Cycles with crossings {1,5,6} will not result in a non-trivial knot.		
{1,5,7}		Crossings 1 and 5 are consecutive non-alternating crossings and, similarly, 5 and 7 are also consecutive non-alternating crossings. Thus, at least one other crossing is required for a non-trivial knot. {1,**2**,5,7}, {1,**3**,5,7}, {1,5,7,**8**} and {1,5,7,**9**} all contain two consecutive non-alternating crossings already covered in Appendix A. {1,**4**,5,7} was covered above. {1,5,**6**,7}: Edge cd is unavailable since

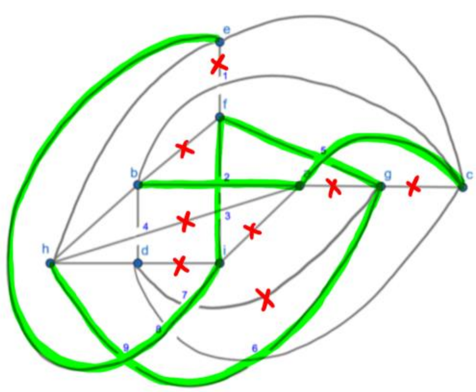
		deg(c)=2.
Cycles with crossings {1,5,7} will not result in a non-trivial knot.		
{1,5,8}		It is not possibly to have a knot with this crossing combination since deg(c)=3.
Cycles with crossings {1,5,8} will not result in a non-trivial knot.		
{1,5,9}		Since crossings 1 and 5 are consecutive non-alternating crossings at least one other crossing is required for a non-trivial knot. {1,**2**,5,9}, {1,**3**,5,9}, {1,5,**7**,9}, {1,5,**8**,9} and {1,5,**6**,9}have two consecutive non-alternating crossings already covered in Appendix A. {1,**4**,5,9} was covered above.
Cycles with crossings {1,5,9} will not result in a non-trivial knot.		

<table>
<tr><td>{1,6,7}</td><td></td><td>These crossings also include 8 & 9 in the cycle, a case already considered in Appendix A.</td></tr>
<tr><td colspan="3">Cycles with crossings {1,6,7} will not result in a non-trivial knot.</td></tr>
<tr><td>{1,6,8}</td><td></td><td>These crossings also include 9 in the cycle. The cycle includes both 8 & 9, a case already considered in Appendix A.</td></tr>
<tr><td colspan="3">Cycles with crossings {1,6,8} will not result in a non-trivial knot.</td></tr>
<tr><td>{2,4,m}</td><td></td><td>Notice that crossings 2 & 4 also include crossing 3 in the cycle. Thus, any crossings {2,4,m} for m=5,6,7,8,9 will include crossings 2 & 3, a case already covered in Appendix A.</td></tr>
<tr><td colspan="3">Cycles with crossings {2,4,m} for m = 5,6,7,8,9 will not result in a non-trivial knot.</td></tr>
</table>

{2,5,6}		Since crossings 2 and 5 are consecutive non-alternating crossings at least one other crossing is required for a non-trivial knot. {**1**,2,5,6}, {2,**3**,5,6},and {2,5,6,**9**}have two consecutive non-alternating crossings already covered in Appendix A. {2,**4**,5,6} was covered above. {2,5,6,**7**}: Edge dg is unavailable since deg(g)=2. {2,5,6,**8**}: After adding edge ei to create crossing 8, this crossing combination will include 8 & 9, a case already considered in Appendix A.
Cycles with crossings {2,5,6} will not result in a non-trivial knot.		
{2,5,7}		Since crossings 2 and 5 are consecutive non-alternating crossings at least one other crossing is required for a non-trivial knot. {**1**,2,5,7}, {2,**3**,5,7}, {2,5,7,**8**} and {2,5,7,**9**} have two consecutive non-alternating crossings already covered in Appendix A. {2,**4**,5,7} and {2,5,**6**,7} were covered above.

<table>
<tr><td colspan="3">Cycles with crossings {2,5,7} will not result in a non-trivial knot.</td></tr>
<tr><td>{2,5,8}</td><td></td><td>Since crossings 2 and 5 are consecutive non-alternating crossings at least one other crossing is required for a non-trivial knot.

{1,2,5,8}, {2,3,5,8}, {2,5,7,8} and {2,5,8,9} have two consecutive non-alternating crossings already covered in Appendix A.

{2,4,5,8} and {2,5,6,8} were covered above.</td></tr>
<tr><td colspan="3">Cycles with crossings {2,5,8} will not result in a non-trivial knot.</td></tr>
<tr><td>{2,5,9}</td><td></td><td>Since crossings 2 and 5 are consecutive non-alternating crossings at least one other crossing is required for a non-trivial knot.

{1,2,5,9}, {2,3,5,9}, {2,5,6,9}, {2,5,7,9} and {2,5,8,9} have two consecutive non-alternative crossings already covered in Appendix A.

{2,4,5,9} was covered above.</td></tr>
<tr><td colspan="3">Cycles with crossings {2,5,9} will not result in a non-trivial knot.</td></tr>
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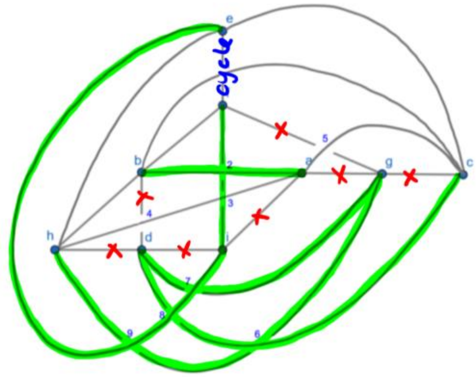
<tr><td>{2,6,7}</td><td>
</td><td>These crossings also include 8 & 9 in the cycle, a case already considered in Appendix A.</td></tr>
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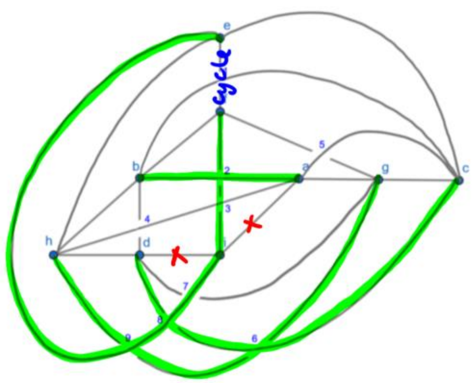
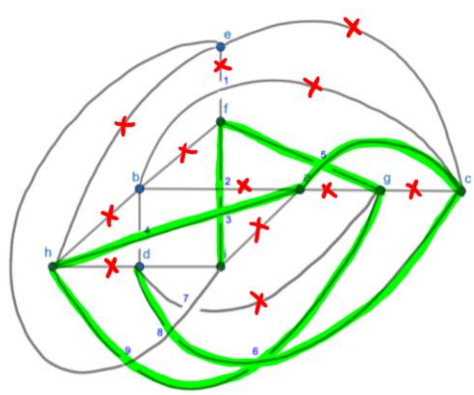
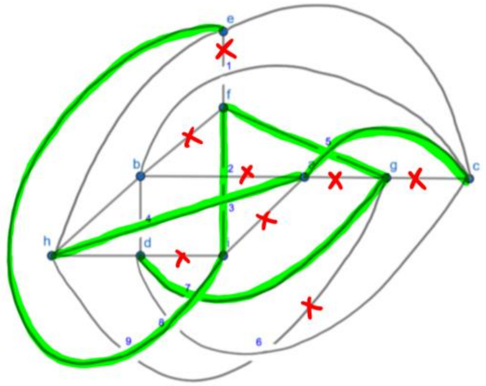
<tr><td>{2,6,8}</td><td>
</td><td>These crossings also include 9 in the cycle. The cycle has crossings 8 & 9, a case already considered in Appendix A.</td></tr>
<tr><td colspan="3">Cycles with crossings {2,6,8} will not result in a non-trivial knot.</td></tr>
<tr><td>{3,5,6}</td><td></td><td>From i, there are two ways to continue. However, since deg(e) = 1, the only option is to add the edge di. This gives a trivial knot.</td></tr>
<tr><td colspan="3">Cycles with crossings {3,5,6} will not result in a non-trivial knot.</td></tr>
</table>

<table>
<tr><td>{3,5,7}</td><td></td><td>Since crossings 3 and 5 are consecutive non-alternating crossings at least one other crossing is required for a non-trivial knot.

{1,3,5,7}, {2,3,5,7}, {3,4,5,7}, {3,5,7,8} and {3,5,7,9} have two consecutive non-alternating crossings already covered in Appendix A.

{3,5,6,7} was covered above.</td></tr>
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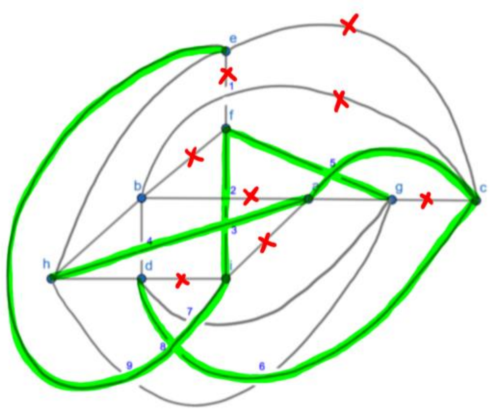
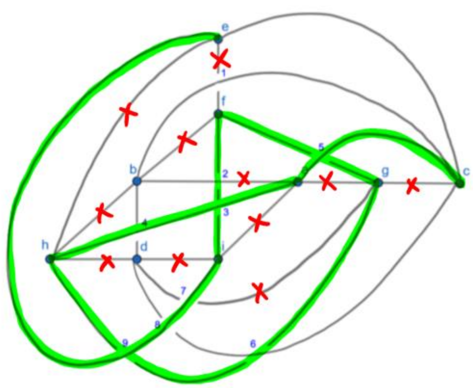
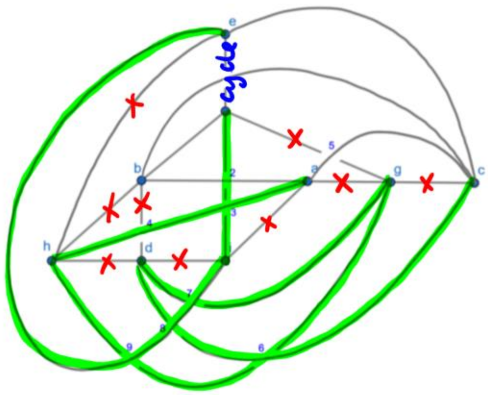
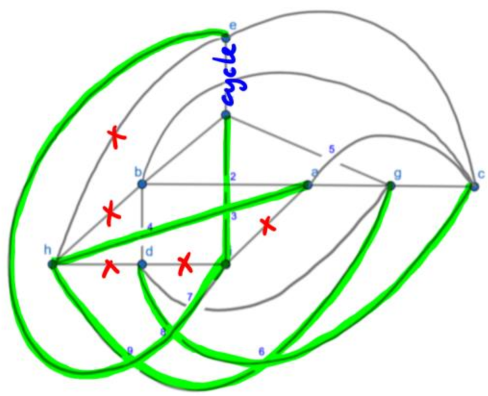
<tr><td>{3,5,8}</td><td></td><td>Since crossings 3 and 5 are consecutive non-alternating crossings at least one other crossing is required for a non-trivial knot.

{1,3,5,8}, {2,3,5,8}, {3,4,5,8}, {3,5,7,8} and {3,5,8,9} have two consecutive non-alternating crossings already covered in Appendix A.

{3,5,6,8} was covered above.</td></tr>
<tr><td colspan="3">Cycles with crossings {3,5,8} will not result in a non-trivial knot.</td></tr>
<tr><td>{3,5,9}</td><td></td><td>Since crossings 3 and 5 are consecutive non-alternating crossings at least one other crossing is required for a non-trivial knot.

{1,3,5,9}, {2,3,5,9}, {3,4,5,9}, {3,5,6,9}, {3,5,7,9} and {3,5,8,9} have two consecutive non-alternating crossings already covered in Appendix</td></tr>
</table>

<table>
<tr><td></td><td></td><td>A.</td></tr>
<tr><td colspan="3">Cycles with crossings {3,5,9} will not result in a non-trivial knot.</td></tr>
<tr><td>{3,6,7}</td><td>
</td><td>These crossings also include 8 & 9 in the cycle, a case already considered in Appendix A.</td></tr>
<tr><td colspan="3">Cycles with crossings {3,6,7} will not result in a non-trivial knot.</td></tr>
<tr><td>{3,6,8}</td><td>
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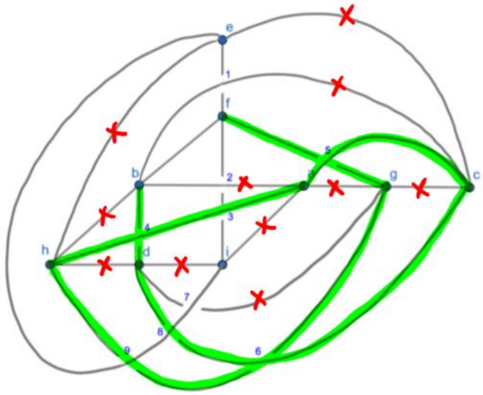
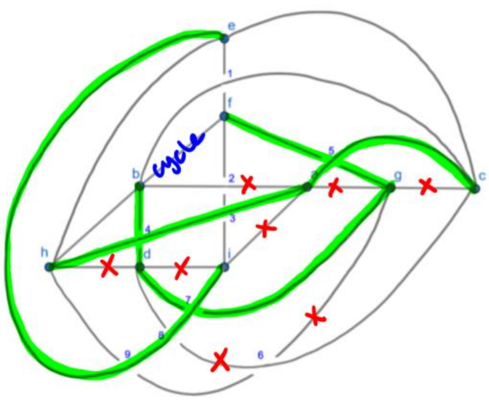
<table>
<tr><td>{4,5,6}</td><td></td><td>Since crossings 5 and 6 are consecutive non-alternating crossings at least one other crossing is required for a non-trivial knot.

{3,4,5,6} and {4,5,6,9} have two consecutive non-alternating crossings already covered in Appendix A.

{1,4,5,6} and {2,4,5,6} were covered above.

{4,5,6,7}: Edge dg is unavailable since deg(d)=deg(g)=2.

{4,5,6,8}: After adding edge ei to create crossing 8, this crossing combination will include 8 & 9, a case already considered in Appendix A.</td></tr>
<tr><td colspan="3">Cycles with crossings {4,5,6} will not result in a non-trivial knot.</td></tr>
<tr><td>{4,5,7}</td><td></td><td>Since crossings 4 and 7 are consecutive non-alternating crossings at least one other crossing is required for a non-trivial knot.

{3,4,5,7}, {4,5,7,8} and {4,5,7,9}have two consecutive non-alternating crossings already covered in Appendix A.

{1,4,5,7}, {2,4,5,7}, {4,5,6,7} were covered above.</td></tr>
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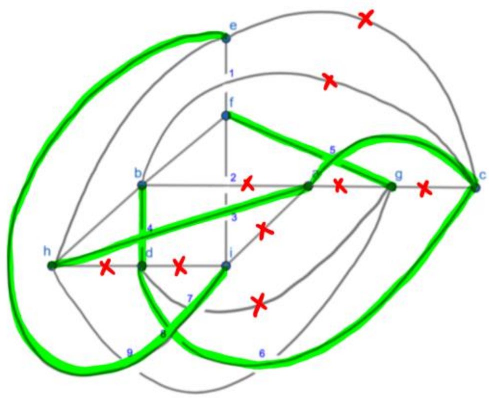
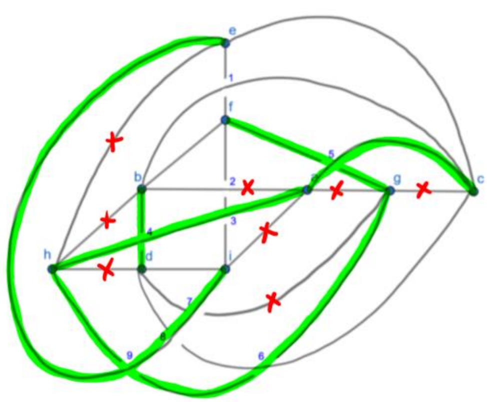
<tr><td colspan="3">Cycles with crossings {4,5,7} will not result in a non-trivial knot.</td></tr>
<tr><td>{4,5,8}</td><td></td><td>Since crossings 4 and 6 are consecutive non-alternating crossings at least one other crossing is required for a non-trivial knot.

{3,4,5,8}, {4,5,7,8} and {4,5,8,9}have two consecutive non-alternating crossings already covered in Appendix A.

{1,4,5,8}, {2,4,5,8}, {4,5,6,8} were covered above.</td></tr>
<tr><td colspan="3">Cycles with crossings {4,5,8} will not result in a non-trivial knot.</td></tr>
<tr><td>{4,5,9}</td><td></td><td>Either crossings 4 and 5 are consecutive non-alternating crossings or else 4, 3, and 5 are three consecutive non-alternating crossings. In either case, at least one other crossing is required for a non-trivial knot.

{3,4,5,9}, {4,5,6,9}, {4,5,7,9} and {4,5,8,9}have two consecutive non-alternating crossings already covered in Appendix A.

{1,4,5,9}, {2,4,5,9} were covered above.</td></tr>
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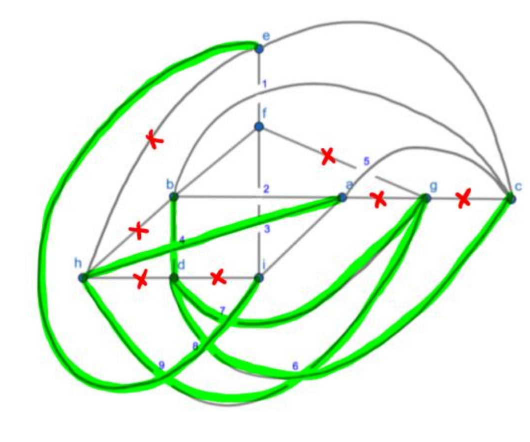
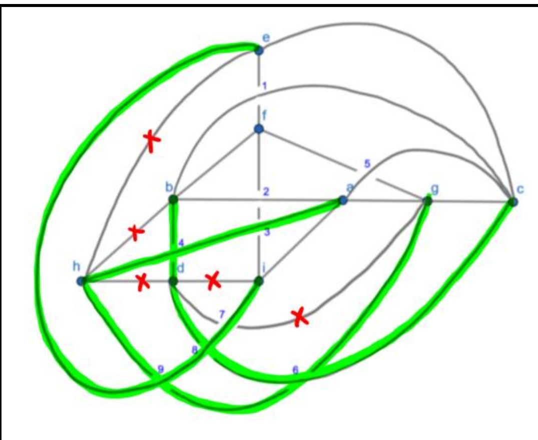
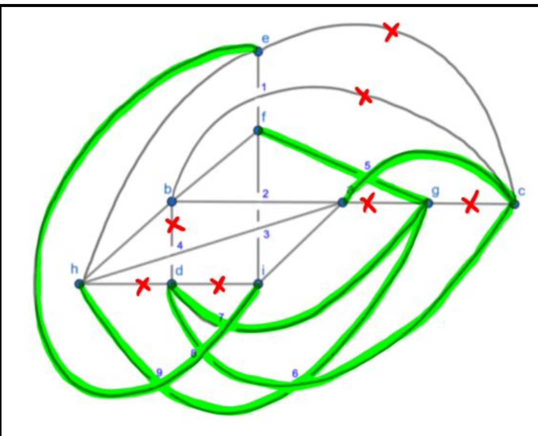
<tr><td>{4,6,7}</td><td></td><td>It is not possibly to have a knot with this crossing combination since deg(d)=3.</td></tr>
<tr><td colspan="3">Cyclese with crossings {4,6,7} will not result in a non-trivial knot.</td></tr>
<tr><td>{4,6,8}</td><td></td><td>These crossings also include 9 in the cycle. The cycle includes crossings 8 & 9, a case already considered in Appendix A.</td></tr>
<tr><td colspan="3">Cycles with crossings {4,6,8} will not result in a non-trivial knot.</td></tr>
<tr><td>{5,6,7}</td><td></td><td>It is not possibly to have a knot with this crossing combination since deg(g)=3.</td></tr>
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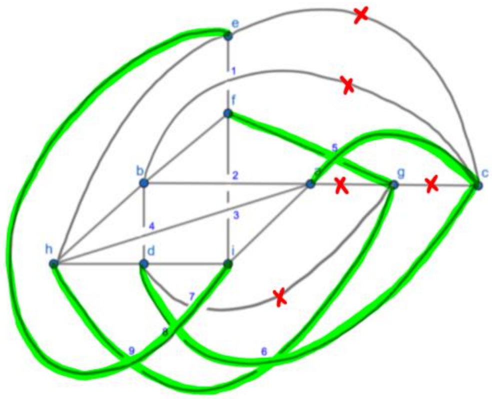
<tr><td>{5,6,8}</td><td></td><td>These crossings also include 9 in the cycle. The cycle includes crossings 8 & 9, a case already considered in Appendix A.</td></tr>
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