Chapter 1
Background and Definitions

Background

In this thesis we will prove a conjecture due to Sachs. We begin by stating the conjecture, giving a brief history of knot theory, and defining terms used in the problem statement. Sachs (1981) observes that for $n = 6$ and $n = 7$, a graph on n vertices with more than $4n - 10$ edges is not discatenable (not discatenable means intrinsically linked, see definition on page 7), and presents a question, "Is it true for all values of n, or are there discatenable graphs on n vertices which have more than $4n - 10$ edges?" In this paper we will answer this question by proving a related statement that we will call Sachs' Intrinsic Linking Conjecture.

Every graph on $n \geq 6$ vertices with $4n - 9$ or more edges is intrinsically linked.

This conjecture combines ideas from both knot and graph theory, so it is important that we know a little bit about both fields. We will assume that the reader has some familiarity with graph theory and will spend much of the following discussion on definitions and history from knot theory.

Brief History of Knot Theory

Johann Listing wrote the book *Vorstudien zur Topologie* in 1847 in which he commented extensively on the study of knots (Devlin, 1998a). In 1867 Lord Kelvin theorized that atoms were knots in ether. Because Kelvin thought that different elements were distinguished by how knotted the ether was, he set out to record the different types of knots. This theory was taken quite seriously, because it seemed to explain quite a few things about matter. By looking at the different types of knots, he was hoping to come up with a type of periodic table and with the help of Peter Tait, they were able to generate substantial knot tables. This idea that atoms were knots in ether was later abandoned but there was sufficient work done on knot theory to build an interest in the field.

Because knots are hard to quantify, it was difficult to come up with any knot invariants. An invariant is a property of a knot that remains unchanged when the projection of the knot is altered. Also because there were no real applications of the study of knots, relatively few mathematicians were drawn to the field of knot theory until the Jones polynomial was discovered in 1984. In addition to spurring interest in polynomial knot invariants, Vaughn Jones was able to show a relation between knot theory and statistical mechanics. The patterns that show up in knot theory and the Jones polynomial also show up in statistical mechanics (Adams, 1994).

Beginning in the early 1980's, knot theory started to appear in the study of viruses (Devlin, 1998b). DNA strands are wound up in knots so that they can fit into a cell nucleus and when a virus attacks, it changes the cell structure and the knottedness of the DNA. By studying the change in the structure of the knot, scientists hope to gain an understanding of how a virus works and eventually come up with a possible cure to viruses like HIV. Even though knot theory did not seem to have any real life applications for a long time, we are starting to see more and more use for its study. In fact, physicists now think that matter is made up of superstrings, which are tiny, knotted closed loops in space-time whose properties are linked to how they are knotted.

Definitions

Livingston (1993) defines a **knot** to be a simple closed curve in R^3. A good way to think of a knot is to take a piece of string or rope and twist, cross and loop it and then connect the ends. When we draw a representation of a knot on a flat surface we call that a **projection**. Because the knot can cross over itself we draw a broken line to represent where the "string" passes under itself. If you take a string and make the first move like you are tying your shoes and then connect the ends, you get a knot called the **trefoil**. The following figure shows two ways that a trefoil can be represented in space.

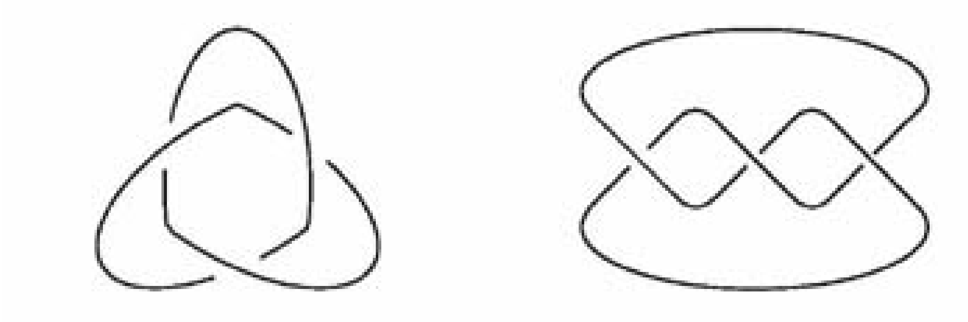


Fig. 1: Two Projections of the Trefoil

Although the projections in Figure 1 represent different curves in R^3, we will think of them as representing the same trefoil knot. Formally, a knot is an equivalence class of curves related by isotopy. Informally, you can change the appearance of the knot by deforming it as much as you like and the knot remains the same knot as long as you do not cut the string or rope. The figure below shows how you can deform a trefoil from one form to another.

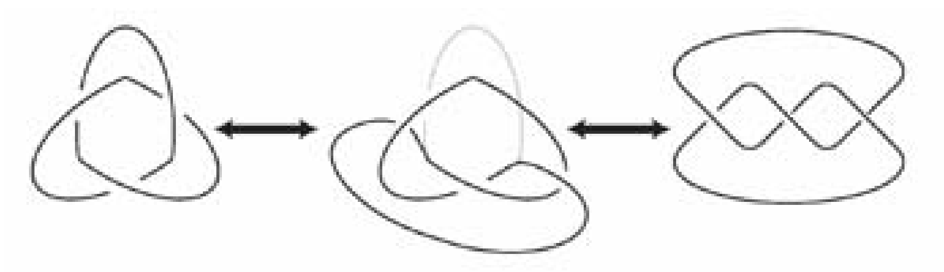


Fig. 2: Deforming a Trefoil From One Projection to Another

If you take a piece of string or rope and connect the ends in just a circle you get the **unknot**. This is the simplest knot and is also called the **trivial knot**.

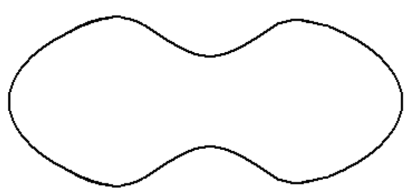


Fig. 3: A Projection of the Unknot

The unknot can also appear in space in different ways besides a circular type shape. The following figure shows the unknot in a different projection from the one above.

(However, it may be difficult at first glance to see that the knot in the next figure will unravel to become the unknot.)

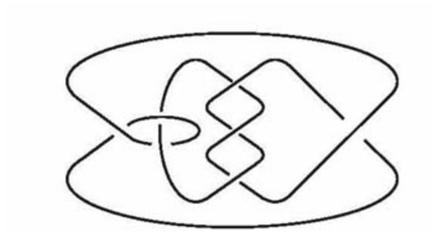


Fig. 4: A More Complicated Projection on the Unknot

A **nontrivial knot** is any knot that cannot be deformed or changed into the unknot without cutting the string. For example, a trefoil is a nontrivial knot because there is no way to eliminate the crossings without cutting the string. (This is intuitively obvious, but quite difficult to prove rigorously.)

If you take two or more knots and tangle them together then you get a **link**. Two links are equivalent if you can change one into the other without cutting any knot or disconnecting the ends. A link that consists of two or more unknots that are not tangled together at all is called a **trivial link** or the **unlink** (e.g. see Fig. 5).

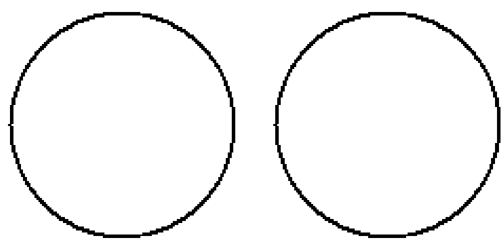
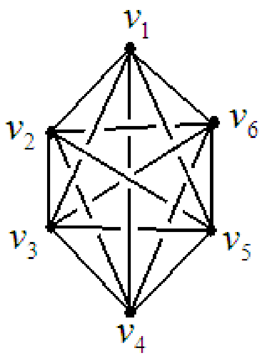


Fig. 5: A Projection of a Trivial Link

A **nontrivial link** is a link that cannot be deformed into an unlink without cutting one or more of the knots that make up the link (e.g. see Fig. 6).

Fig. 6: An Example of a Nontrivial Link

In trying to understand more about knots and their behavior, a connection has been made between knots and links and graphs. A **graph** is defined by a set of vertices (or points) and the edges (or arcs) that connect the vertices. We will label the vertices of the graphs with lower case letters such as *a* and *b*, and $\{ab\}$ will denote the edge between vertices *a* and *b*. $V(G)$ will denote the set of all the vertices of *G* and $|V(G)|$, the total number of vertices. $E(G)$ will denote the set of all the edges in graph *G* and $|E(G)|$, the total numbers of edges. Like knots, we look at graphs in three-dimensional space (R^3) so that edges can pass over and under one another. Because we cannot show three dimensions on paper, when a graph is drawn, a broken line will show an edge that passes behind another edge. A **complete graph** is a graph where each vertex is connected to every other vertex by an edge. K_n denotes a complete graph on *n* vertices, so K_6 would represent a complete graph on 6 vertices. An **embedding** is a particular way of placing a graph in space. Vertices are represented by disjoint points, and edges, by arcs joining the vertices. Edges can intersect one another only at the vertices. A graph can have lots of different embeddings that may not look at all alike. Figure 7 shows two different embeddings of K_6.



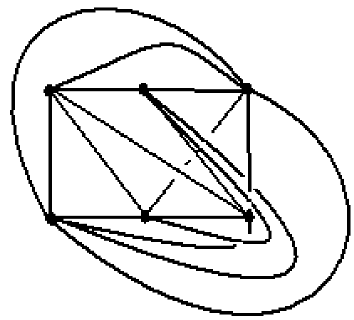


Fig. 7: Two Different Embeddings of K_6

Now, because we look at knots and graphs in space, an embedding of a graph can also be knotted or have links in it. For linking in a graph we can, for example, look at triangles in the graph. A triangle is formed when we take any three vertices and connect them with existing edges. (Note that we are using "triangle" in the topological sense. In particular, the edges need not be line segments.) For example, if we take v_1, v_2, v_3 and then v_4, v_5, v_6 from the graph of K_6 from the left in Figure 7 and connect them with the existing edges then we get the following two triangles.

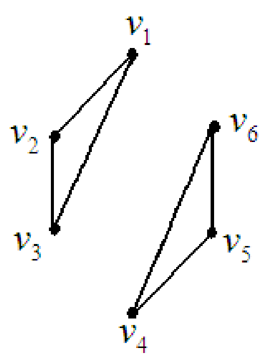


Fig. 8: Two Unlinked Triangles

Notice that the two triangles are not linked, but if we take v_1, v_2, v_5 and then v_3, v_4, v_6 from the graph of K_6 and connect them with the existing edges then we get the following two triangles that are linked together.

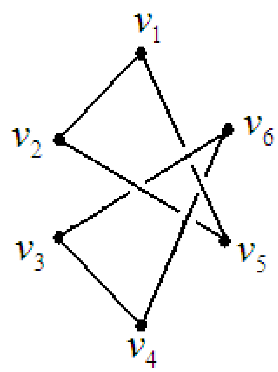


Fig. 9: Two Linked Triangles From K_6

In their paper *Knots and Links in Spatial Graphs*, Conway and Gordon (1985) showed that every embedding of K_6 contains a nontrivial link like the one shown above. We will also say that the graph K_6 is **intrinsically linked**. This means that no matter how we embed K_6 in space, there will always be at least one pair of polygons that are linked together.

A graph is said to be **planar** if there is an embedding of it that can be completely contained in a plane. So, if we can draw the graph on a flat surface without having any edges cross one another then the graph is planar. A graph is **nonplanar** if there is no embedding of the graph that can be contained in a plane. For example, K_4 is a planar graph but K_5 is non-planar as illustrated in the embeddings below.

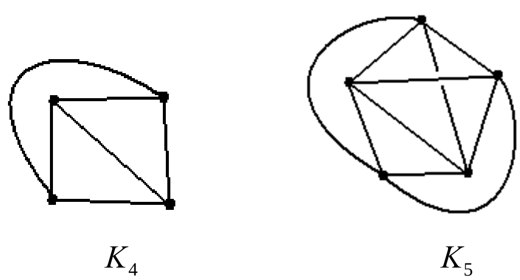


Fig. 10: Embeddings of K_4 (Planar) and K_5 (Nonplanar)

In fact, while Figure 10 does show that K_4 is planar, it does not “prove” that K_5 is not planar. The proof that K_5 is not planar is part of Kuratowski’s Theorem.

Kuratowski's Theorem: A graph G is nonplanar if and only if it has $K_{3,3}$ or K_5 as a minor.

$K_{3,3}$ is a graph (an embedding is shown in Fig. 11) that has two sets containing 3 vertices $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ such that each vertex in the first set is connected by an edge to each vertex in the second set, but no vertex is connected to any vertex in its own set. The vertex v_1 is connected to v_4, v_5, and v_6 with an edge, but it is not connected to v_2 or v_3.

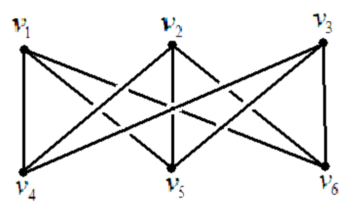


Fig. 11: An Embedding of $K_{3,3}$

We say G' is a minor of G if G' is obtained from G by eliminating edges or vertices or contracting edges (see definition in Chapter 2).

Our approach to Sachs' Intrinsic Linking Conjecture will be to argue that such a graph has K_6 as a minor. It then follows from Conway and Gordon's work that the graph is intrinsically linked.

In the next chapter, we will motivate the conjecture using Euler's formula $V + F = E + 2$. We will also discuss a proof about triangle-Y exchanges and define more terms that will be important to the development of our proof. In Chapter 3, we will prove Sachs' Intrinsic Linking conjecture and finally in Chapter 4, we will summarize and suggest some open problems that remain.

This proof was part of a collaborative effort with Sam Williams and Matthew Rodrigues as part of a REU/T program in the summer of 2005 at Chico State University. This was a continuation of work started in the summer of 2003 and was presented in a paper by Campbell, Mattman, Ottman, Pyzer, Rodrigues, and Williams (Campbell et al.,

2005) about intrinsic linking and knotting of graphs. We also discovered that Mader published a proof in 1968 that showed a graph on n vertices with at least $4n - 9$ edges has a K_6 minor, which implies Sachs' Intrinsic Linking Conjecture. Although the conjecture was already proven, the proof presented here is much different from Mader's.

Chapter 2
Motivation for Proof

In this chapter, we will use Euler's Formula to motivate Sachs' Intrinsic Linking Conjecture. From high school geometry we know **Euler's formula** and how it relates to polyhedra. Recall that it states that the sum of the number of vertices and faces of any polyhedron that is topologically equivalent to a sphere is equal to 2 more than the number of edges, more commonly written as $V + F = E + 2$. Euler's formula is also valid for connected planar graphs. If you notice the embedding of K_4 in Figure 10 above, it has 4 vertices, 4 faces because we count the outside of the graph as a face, and 6 edges. Substituting into Euler's formula, we get $4 + 4 = 6 + 2$, which as we can see is true.

All the graphs that we will consider are **simple graphs**, which means that there can only be one edge between any two vertices and there are no loops or edges that connect a vertex to itself. Because this is the case, then the minimum number of edges that can enclose a face on a graph is three. We can use this relationship to find an upper bound for the number of edges that there can be on any connected planar graph. So on any connected planar graph, because there are at least three edges for each face, then the number of edges is at least 3 times the number of faces. But each edge is part of two faces so the number of edges is actually at least half of 3 times the number of faces ($E \geq \frac{3F}{2}$).

We can rearrange Euler's formula into $F = E - V + 2$ and rearrange $E \geq \frac{3F}{2}$ into $F \leq \frac{2E}{3}$ and then use substitution to get a relationship between the number of edges and vertices of a connected planar graph.

$$F = E - V + 2$$
$$E - V + 2 \leq \frac{2E}{3}$$
$$3E - 3V + 6 \leq 2E$$
$$E \leq 3V - 6$$

So we have an upper bound for the number of edges that a connected graph can have and still be planar. If a connected graph is planar then the number of edges is no more than six less than three times the number of vertices. This is also an upper bound for a graph that is not connected, since such a graph can be made connected by adding edges.

A **triangle –Y exchange** is where the three edges of a triangle in a graph are replaced by a new vertex v_n and three edges. A **Y-triangle exchange** is the opposite: Three edges and a vertex are replaced by a triangle. The following is an example of a triangle-Y exchange (at left) and a Y-triangle exchange (at right).

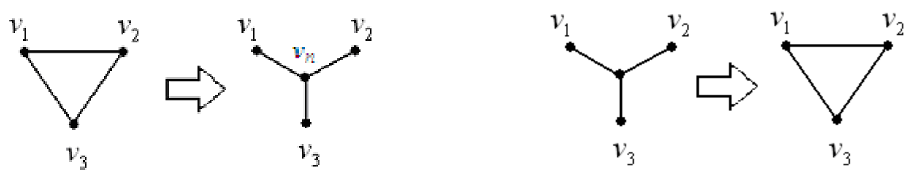


Fig. 12: A Triangle-Y and Y-Triangle Exchange

Motwani, Raghunathan, and Saran (1988) showed that triangle – Y exchanges preserve intrinsic linking in a graph. A summary of their proof follows.

Suppose that a graph G' is formed by performing a triangle – Y exchange on *G*. We want to show that if the graph *G* cannot be embedded without being linked, then G' also cannot be embedded without being linked. We want to prove this by contradiction so we are going to assume that G' can be embedded without being linked. Because we want to see if the triangle – Y exchange affects whether or not the graph is linked, we only need to look at the vertices and edges that are part of the triangle – Y exchange.

We will take a projection of G' near *x* (see Fig. 13) and deform that portion from how it appears on the left to how it appears on the right. By first moving the vertices *u*, *v*, and *w* along their edges toward *x*, we can assume that there are no other edges crossing the “Y” and the projection is exactly as shown near *x*. This change does not affect the linking structure of G'. We will say that the triangle on the right represents part of an embedding of *G* that we want to show has no two cycles that are linked.

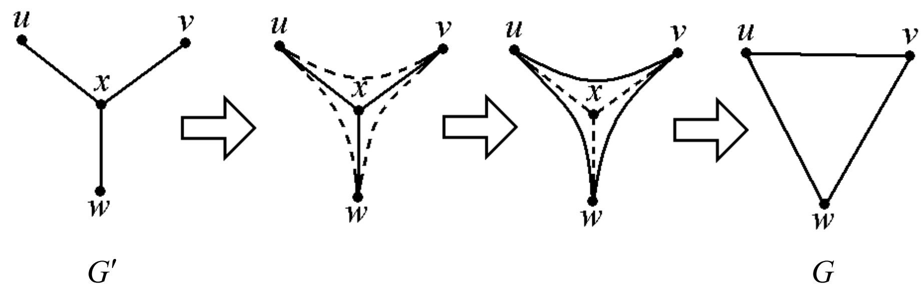


Fig. 13: Deforming the Y to the Triangle

Let's assume instead that there are two cycles C_1 and C_2 that are linked together in G. The triangle uvw is not linked because it is not part of any crossings in G. Also because G' did not have a pair of linked cycles then at least one of the edges of uvw must be used in the link. Let's say that C_1 uses these edges and the link enters at u and leaves through v. C_1 can either use edge $\{uv\}$ or it can use the two edges $\{uw\}$ and $\{wv\}$. The corresponding cycle in G' would use the edges $\{ux\}$ and $\{xv\}$ to get from u to v. So the cycles C_1 and C_2 can be represented by corresponding cycles C_1' and C_2' in G'. But there is no additional crossing introduced, so since C_1' and C_2' are not linked, C_1 and C_2 also cannot be linked. We have a contradiction; so if G' is not linked then G cannot be linked. In other words, if G is intrinsically linked, then G' is too.

Robertson, Seymour and Thomas (1993) showed that a graph is intrinsically linked if and only if it has one of the seven Petersen graphs (shown in Fig. 14) as a minor. The **Petersen graphs** are formed by performing repeated triangle –Y and Y-triangle exchanges on K_6. (K_6 is the second from the left on the top row of Fig. 14.)

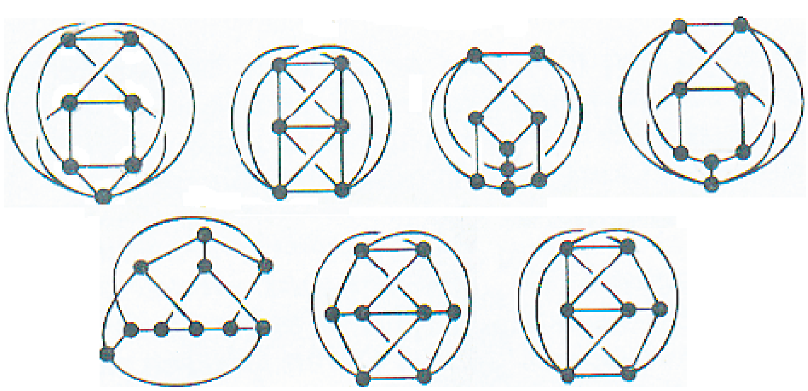


Fig. 14: The Seven Petersen Graphs

A graph G_s is a **subgraph** of G if all of the edges and vertices of G_s are in G. An **induced subgraph** of G is defined by a set of vertices of G as well as all the edges of G that are incident to two of the vertices of that set of vertices. A **minor** of a graph is a graph G_m that can be obtained from a graph G by a series of vertex deletions, edge deletions or edge contractions. A **vertex deletion** is when a vertex and all the edges incident to it are deleted from the graph G and an **edge deletion** is when just an edge is removed from the graph G. An **edge contraction** however is when two vertices, say v_1 and v_2 (see Fig. 15) are combined into one new vertex v_n and every vertex that was connected to v_1 or v_2 by an edge is now connected to v_n by an edge. For example the graph G_m is a minor of graph G because G_m was produced by contracting the edge between v_1 and v_2 to form the vertex v_n. Note that, if G' is a subgraph of G, then G' is also a minor of G since G' can be obtained by vertex or edge deletions.

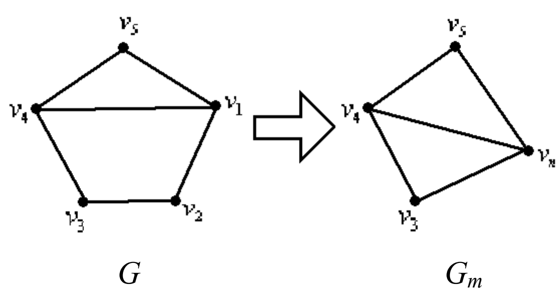


Fig. 15: Edge Contraction of $\{v_1v_2\}$

Sachs (1981) showed:

Theorem 1: If $G = K_1 + H$ then G is intrinsically linked if and only if H is non-planar.

The notation $K_1 + H$ means that we take H and add a new vertex v_n and then connect all of the vertices in H to v_n with new edges, as shown below (Fig. 16).

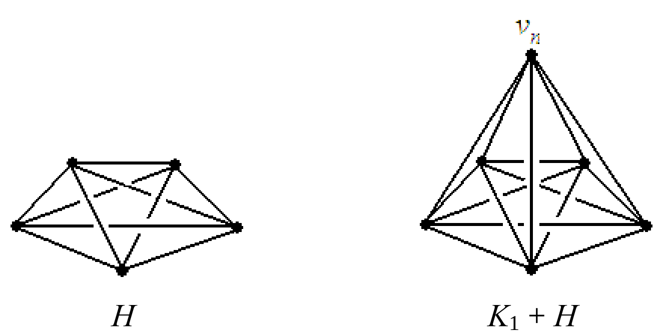


Fig. 16: K_1 Added to a Nonplanar Graph

Now we can use Theorem 1 presented by Sachs and Euler's theorem to come up with Sachs' Intrinsic Linking Conjecture. If we have a planar graph H on $n - 1$ vertices, then by Euler's theorem it can have at most $3(n - 1) - 6$ edges or $E \leq 3(n-1)-6$. If we add K_1 to H, we have a graph G with n vertices and $n - 1$ more edges because the new vertex has to connect to the $n - 1$ existing vertices of H. So we get an upper bound for the number edges of G as $E \leq 3(n-1)-6+(n-1)$ or when we simplify the right side of the inequality $E \leq 4n-10$. Graph G is not intrinsically linked because H is planar, but this does seem to imply that we have an upper bound for the number of edges that a graph could have and possibly not be intrinsically linked. Seeing this relationship, Sachs presents his conjecture as: Are there any non-intrinsically linked graphs on n vertices that have more than $4n-10$ edges $(E > 4n-10)$? In the next chapter we will prove a reformulation of this:

Sachs' Intrinsic Linking Conjecture: Every graph on $n \geq 6$ vertices with $4n-9$ or more edges is intrinsically linked.

Although the reasoning above suggests this conjecture, it is not a proof. Rather it shows that there are many graphs with $4n - 10$ edges that are not intrinsically linked. In this sense, the bound of $4n - 9$ of the conjecture is the best possible.

Chapter 3
The Proof

In this chapter we will prove:

Sachs' Intrinsic Linking Conjecture: Every graph on $n \geq 6$ vertices with $4n-9$ or more edges is intrinsically linked.

We will begin by discussing our first approach to the proof. In Conjecture 1 below, we make a connection between Sachs' Intrinsic Linking Conjecture and the number of common neighbors shared by any pair of adjacent vertices.

Conjecture 1: Let G be a simple, connected graph such that every pair of adjacent vertices has at least four common neighbors. Then G has a K_6 minor.

Proposition 1 shows that Conjecture 1 proves Sachs' Intrinsic Linking Conjecture.

Proposition 1: Conjecture 1 implies Sachs' Intrinsic Linking Conjecture.

Before proving Proposition 1, we must first present and prove a lemma that will support our inductive argument. The lemma requires the following definition. Let $CN(xy)$ be the **set of common neighbors** of two adjacent vertices x and y in graph G. That is $CN(xy) = \{v \in V(G) | \{vx\}, \{vy\} \in E(G)\}$.

Lemma 1: Let $n > 6$ and suppose every graph on $n-1$ vertices with at least $4(n-1)-9$ edges has a K_6 minor. Let G be a graph with n vertices, with $|E(G)| \geq 4n-9$, and vertices u and v such that $|CN(uv)| \leq 3$. Then G has a K_6 minor.

Proof of Lemma 1

Suppose every graph on $n - 1$ vertices with at least $4(n-1)-9$ edges has a K_6 minor. Let *G* be a graph with *n* vertices and at least $4n-9$ edges, with vertices *u* and *v* such that $|CN(uv)| \leq 3$. Since $|CN(uv)| \leq 3$, contracting along edge {*uv*} will remove at most 4 edges from the graph. Since *G* has at least $4n-9$ edges, $G \setminus \{uv\}$ (that is graph *G* with edge {*uv*} contracted) has at least $4n-9-4=4n-13=4(n-1)-9$ edges implying that $G \setminus \{uv\}$ has a K_6 minor. Therefore *G* has a K_6 minor.

(Lemma 1) ~

The **degree**, $\delta(v)$, **of a vertex** *v* is the number of edges connected to *v*. The **degree**, $\delta(G)$, **of a graph** *G* is the least degree among its vertices. Note that the common neighbor condition of Lemma 1 can be replaced with the requirement $\delta(G) \leq 4$. That is, if every vertex is connected to at most 4 other vertices with an edge, then two adjacent vertices can never have more than three common neighbors.

Proof of Proposition 1

We proceed by induction on *n*, the number of vertices of graph *G*. Assume Conjecture 1. If $|V(G)| = 6$, then $|E(G)| = 4(6)-9=15$. The only graph on six vertices and 15 or more edges is K_6. So $G = K_6$, which is intrinsically linked by Conway and Gordon's Theorem. Assume Conjecture 1 implies Sachs' conjecture for any graph *L* with $6 \leq |V(L)| \leq n-1$. Let *G* have *n* vertices. We may assume $|E(G)| = 4n-9$, since if $|E(G)| > 4n-9$, then *G* has a minor with $4n - 9$ edges. If there exists $\{uv\} \in E(G)$ with $|CN(uv)| < 4$, then by Lemma 1 *G* has a K_6 minor, and thus is intrinsically linked. If every pair of adjacent vertices has four or more common neighbors, then by Conjecture 1, *G* has a K_6 minor and is therefore intrinsically linked.

(Proposition 1) ~

Although, as mentioned below, we found evidence that supports Conjecture 1, we were unable to prove it. (We will discuss Conjecture 1 further in Chapter 4). Instead, we will present a more direct proof. The following theorem proves Sachs' Intrinsic Linking Conjecture because any graph that has a K_6 minor is intrinsically linked. The proof incorporates the idea of common neighbors discussed above. We know that if $n \geq 4$ and the number of edges is at least $3n - 5$, then the graph has a K_5 minor (see Bollobás, 1978, Corollary 1.13), but we are going to take this a step further. (Note that the following theorem is also proved, in a different way, by Mader, 1968.)

Main Theorem: If G is a graph with $n \geq 6$ vertices and $4n - 9$ or more edges, then G has a K_6 minor.

Proof

We proceed by using induction on n, the number of vertices of our arbitrary graph G, with $|E(G)| \geq 4n - 9$. Let $n = 6$. Then G has at least $4(6) - 9 = 15$ edges. So G must be K_6, implying that G has a K_6 minor.

Now, suppose $n > 6$. We assume that any graph L on 6, 7, …, $n - 1$ vertices with at least $4|V(L)| - 9$ edges has a K_6 minor. Let G be a graph with n vertices that has $4n - 9$ edges. Note that if $|E(G)| > 4n - 9$, then G has a minor on $4n - 9$ edges which can be obtained through edge deletions.

First suppose that G has two adjacent vertices with less than four common neighbors. Since G has $4n - 9$ edges, G has a K_6 minor by Lemma 1.

So we can assume every pair of adjacent vertices in G has at least four common neighbors. (Note that the remainder of the proof can be thought of as partial verification of Conjecture 1). We can also eliminate degree 0 vertices. If *G* has a degree 0 vertex, then by removing it, we obtain a subgraph that has a K_6 minor by the inductive hypothesis.

This implies that every vertex *v* in *G* has a degree of at least five, because as soon as one edge {*uv*} is incident to *v* there will be at least four more incident to the common neighbors of *u* and *v*. We will focus on a vertex of minimal degree. That is, we will now consider a vertex *a*, with $a \in V(G)$ and such that, for all $v \in V(G), \deg(a) \leq \deg(v)$.

Suppose *G* has such a vertex *a* of degree five. Consider the induced subgraph G' on the neighbors of *a*. Each of the neighbors of *a* must share at least four common neighbors with it, and since G' has 5 vertices and is the graph consisting of every vertex in *G* that is adjacent to *a*, the degree in G' of each vertex in G' must exactly four. But notice that G' is a simple graph with five vertices and $\frac{5(4)}{2} = 10$ edges. There is only one graph on five vertices with 10 edges, so G' must be this graph, K_5. Thus, because *a* is connected to every vertex in G', the induced subgraph on *a* and its neighbors is K_6, implying that *G* has a K_6 minor.

Now suppose $\deg(a) = 6$. Consider the induced subgraph G' on the neighbors of *a*, v_1, v_2, v_3, v_4, v_5, and v_6. Since every vertex v_i must share at least four common neighbors with *a*, then for all $v_i \in \{v_1, v_2, v_3, v_4, v_5, v_6\}, \deg_{G'}(v_i) \geq 4$. That is, the degree of each vertex in the graph G' is at least 4. However, we know that if $|E(G')| \geq 3n - 5 = 3(6) - 5 = 13$, then G' has a K_5 minor (Bollobás, 1978, Corollary 1.13) implying that *G* has a K_6 minor. Since every vertex in G' must have degree at least 4, there are at least 12 edges in G', and we need only consider the case when $|E(G')| = 12$. There is only one graph on 6 vertices and 12 edges with each vertex of degree 4: the degree 4-regular triangulation on six vertices shown in Figure 17. (To see that there is only one such graph, note that it is obtained from K_6 by removing 3 edges such that each vertex is left with degree 4.) Each vertex has a degree of 4 and each face of the graph is topologically equivalent to a triangle.

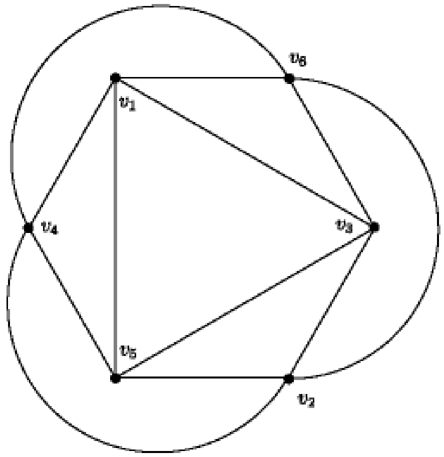


Fig. 17: The Degree 4-regular Triangulation on Six Vertices

Now, adding an edge to G' will make it have 13 edges, implying it has a K_5 minor. On the other hand there are more edges in *G* incident on the vertices of G', as all pairs of adjacent vertices must have at least four common neighbors. So any two adjacent vertices of G' have at least one more common neighbor beyond *a* and the two in G'. For each triangle in G', we will construct a subgraph H_i ($i = 1,2,\ldots,8$) of *G* (see Fig. 18).

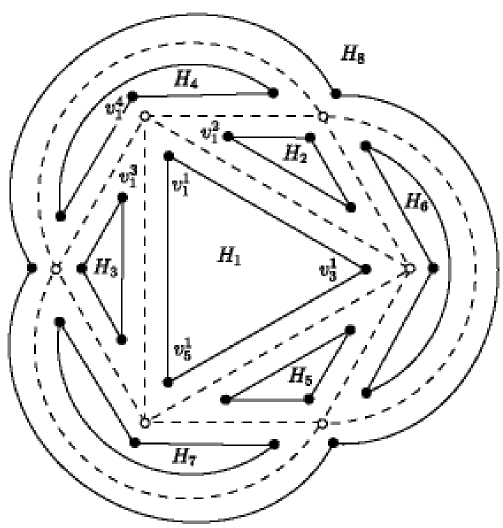


Fig. 18: H_i Subgraphs of G'

Each H_i is induced by three vertices of G' along with additional vertices of G. Each H_i is disjoint from the others, so we will make three copies of each vertex and two copies of each edge. This is because each vertex is part of 3 H_i's and each edge is part of 2 H_i's. Each vertex in $W = V(G) \setminus \{a, v_1, v_2, ..., v_6\}$ will be in exactly one of the H_i. Any vertex not in the connected component of a is part of H_1. Every other vertex w of W is connected to a and, therefore, to at least one of the v_i's by a path that, except for its endpoint v_i, includes none of the vertices of G'. Call such a path **G′-avoiding**. We can assume that no vertex in W can be connected to both v_1 and v_2 by G'-avoiding paths because then we would have a path from v_1 to v_2 that was not in G'. We could edge contract along this path and produce an edge $\{v_1 v_2\}$ that is not in G'. This would make a minor of G on $v_1, v_2, \ldots, v_6$ that has more than 12 edges. Then, together with a, this would constitute a K_6 minor for G. Similarly $w \in W$ cannot connect to both v_3 and v_4, nor to both v_5 and v_6.

So each vertex w in W is connected to at most 3 vertices of G' by G'-avoiding paths. If w is connected to 3 vertices of G', then w is a vertex of the corresponding H_i. For example, if w is connected to all of v_1, v_3, and v_5 by G'-avoiding paths, then w is a vertex of H_1 (see Fig. 18). If w is connected to two vertices of G', then w is a vertex in the H_i of lowest index that contains the edge between the two vertices that w is connected to. For example, a vertex connected to v_1 and v_3 by G'-avoiding paths is placed in H_1 even though H_2 is also defined by v_1 and v_3 (along with v_6). If w is connected to one vertex of G', then it is placed in the H_i of lowest index. For example, if a vertex connected to v_2 by G'-avoiding paths then it is placed in H_5. If a vertex is not connected to any vertex of G' by G'-avoiding paths then it is placed in H_1. Note that if there is a G'-avoiding path between two vertices in W, then the two vertices are of the same type. That is, they are both connected to the same 0, 1, 2, or 3 vertices in G'. In this way, we construct the induced subgraphs H_i so that (we may assume) there is no path in G, disjoint from the vertices of G', that connects any two H_i and H_j, since such a path would allow us to produce an additional edge, like $\{v_1 v_2\}$, and hence a K_6 minor of G. This means that having distributed the vertices of W among the H_i, the induced subgraphs will include every edge of G except those incident to a.

Let G'' be the disjoint union of the H_i as illustrated in Figure 18. We will now count the edges of G''. There are $4n-9$ edges in G, and a is not in G'', so we must subtract its six edges. Each edge of G' is in two subgraphs H_i and H_j so we must add 12 more edges. Thus, $|E(G'')| = 4n-9-6+12 = 4n-3$. However, G'' does not have n vertices, or at least it does not have the same number of vertices as G, so we must count the vertices in G''. There are n vertices in G. We exclude a, but each vertex of G' is in 4 subgraphs so we must add $3(|V(G')|) = 3(6) = 18$. (e.g. Fig. 18 shows that v_1 will appear in H_1, H_2, H_3, and H_4 as suggested by the superscripts.) Counting thus reveals that G'' has $m = n-1+18 = n+17$ vertices, and if we substitute $m-17$ in for n then we get $4n-3 = 4(m-17)-3 = 4m-4(17)-3 = 4m-71$ edges in G''.

Since G'' is the union of the H_i, the number of vertices and edges of G'' is the sum of the number of vertices and edges of each H_i. So we let $|E(H_i)| = 4m_i - k_i$ where $m_i = |V(H_i)|$, $m = m_1 + m_2 + \cdots + m_8$ and $71 = k = k_1 + k_2 + \cdots + k_8$. Suppose each $k_i > \left\lfloor \frac{k}{8} \right\rfloor$. (Of course $\left\lfloor \frac{k}{8} \right\rfloor = 8$, but we will use the notation $\left\lfloor \frac{k}{8} \right\rfloor$ to suggest how the argument can be modified for the $\deg(a) = 7$ case below.) If $k_i > \left\lfloor \frac{k}{8} \right\rfloor$, then $k_i \geq 9$. Therefore:

$$\sum k_i \geq 72$$

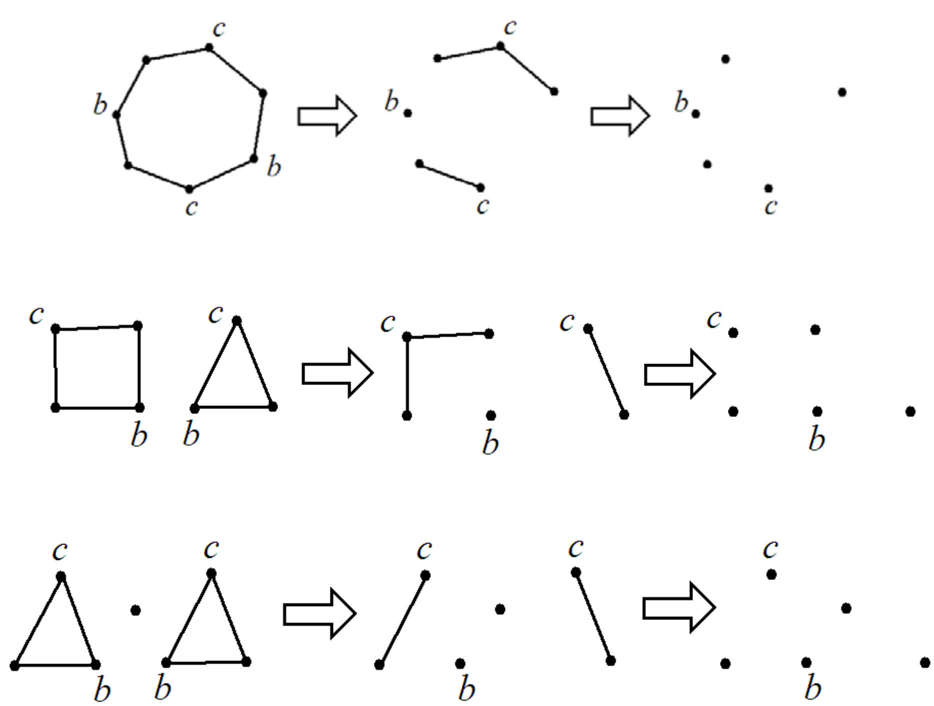
$$-\sum k_i \leq -72$$

$$4\sum m_i - \sum k_i \leq 4\sum m_i - 72$$

$$\sum (4m_i - k_i) \leq 4m - 72$$

$\sum (4m_i - k_i) \leq 4m - 72 < 4m - 71$ which is a contradiction. So there is a k_i with $k_i \leq \left\lfloor \frac{k}{8} \right\rfloor$. Hence there is an H_i where $|E(H_i)| \geq 4m_i - \left\lfloor \frac{k}{8} \right\rfloor = 4m_i - 8$. Now $m_i \geq 3$ because H_i contains at least the three vertices from G', and $|E(H_i)| \geq 4(3) - 8 = 4$. So H_i cannot be a simple triangle and includes at least one vertex from W. That vertex has degree 6 or more, so $m_i \geq 7$ (i.e. that vertex, along with all its neighbors, must be in H_i). We can also get an upper bound for m_i because, if we omit the other 7 H_i's, we will get at least 7(3) = 21 fewer vertices than we had in G''. So we can say $7 \leq m_i \leq n-4$. Thus G has a minor, H_i, on $m_i < n$ vertices with more than $4m_i - 9$ edges, implying that H_i has a K_6 minor by our induction assumption. Therefore G has a K_6 minor.

The argument when G has a vertex of degree 7 is quite similar. We know that if $|E(G')| \geq 3n-5 = 3(7)-5 = 16$ then G' contains a K_5 minor and thus G contains a K_6 minor. There are 8 graphs on 7 vertices of degree at least 4 and having fewer than 16 edges (Here we are removing 6 or 7 edges from K_7 so that each vertex has degree 4 or more.), but only one of these has no K_5 minor. To show this we will look at the complements of these 8 graphs. So we need to consider the graphs on 7 vertices of degree at most 2 and having 6 or 7 edges. In Figure 19, the edge contractions and one vertex deletion (in the 6th graph) are shown that would lead to a K_5 minor. We will contract the edge between the b vertices in the first step and then the edge between the c vertices in the second to obtain the complement of K_5.



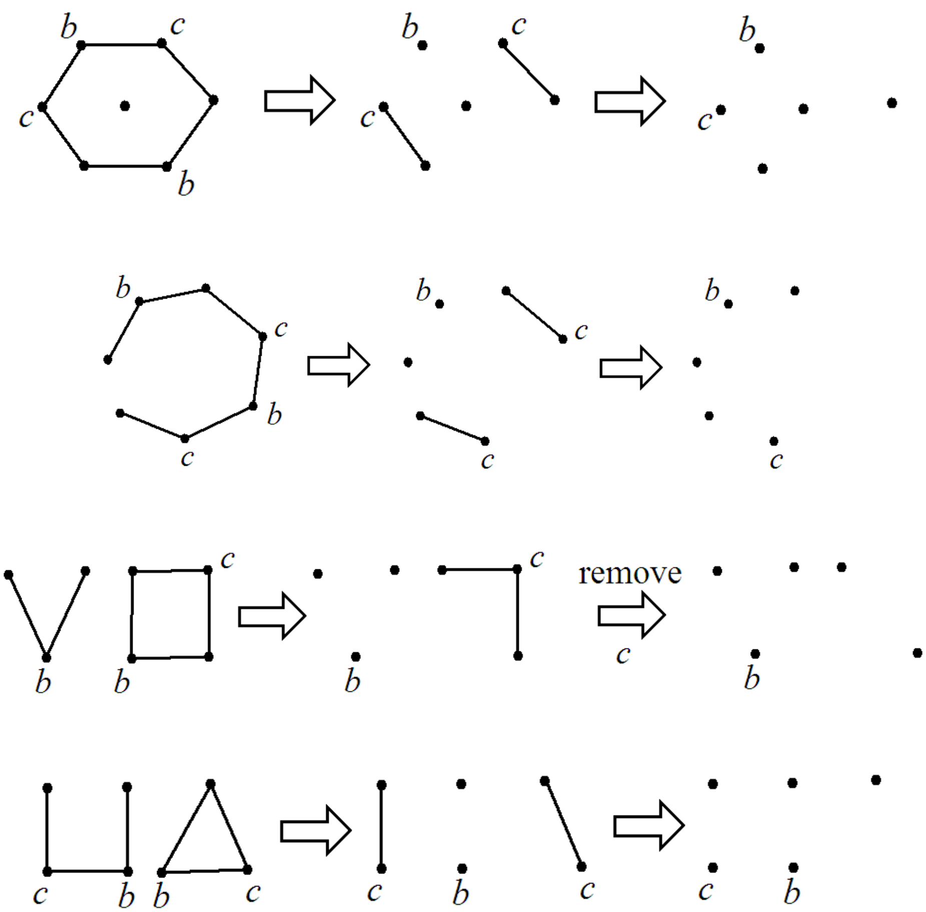


Fig. 19: K_5 Minors in Our Graphs With 7 Vertices

The graph that does not have a K_5 minor is a triangulation on seven vertices with two vertices of degree five and five vertices of degree four, and 10 triangles (Fig. 20). Note, we will also define the H_i's and G'' the same as in the previous case. In this case G'' has $m = n + 22$ vertices and $4m - 89$ edges. There must be a subgraph of G'' that has m_i vertices and at least at $4m_i - \left\lfloor \frac{89}{10} \right\rfloor = 4m_i - 8$ edges. So G has a subgraph that has a K_6 minor, implying that G has a K_6 minor.

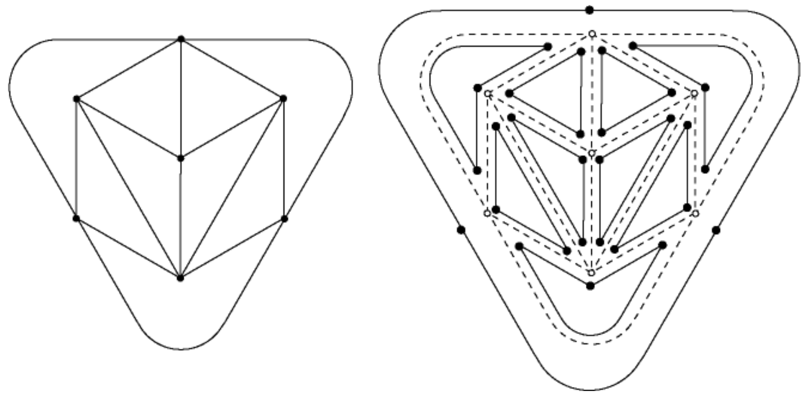


Fig. 20: The Triangulation on 7 Vertices

The only other possibility is that every vertex of G has degree greater than or equal to 8. However, notice that $\delta(G) \geq 8$ implies that G has at least $\frac{8n}{2} = 4n$ edges, which contradicts our assumption that G has exactly $4n - 9$ edges. Thus $\delta(G) \geq 8$ is not possible.

(Theorem) ~

Notice that early in the proof we were able to assume that every pair of adjacent vertices in G has four common neighbors. This suggests that our proof could be modified to show Conjecture 1. We will return to this idea in the next chapter.

Our theorem proves Sachs' Intrinsic Linking Conjecture since any graph that has a Petersen graph, like K_6, as a minor is intrinsically linked.

Chapter 4
Overview and Future Research

In this chapter we will summarize our results and discuss some ideas for future research.

Sachs (1981) showed that if $G = K_1 + H$ then G is intrinsically linked if and only if H is nonplanar. From Euler's Formula about polyhedra, we found that the maximum number of edges that a planar graph with n vertices can have is $3n - 6$. This gave us a lower bound for the number of edges that would guarantee a nonplanar graph as $3n - 6 + 1$ or $3n - 5$. Now when we added a K_1 to a nonplanar graph on $n - 1$ vertices, we add $n - 1$ edges which gives us $3(n - 1) - 5 + n - 1$ edges or $4n - 9$ edges. This led us to:

Sachs' Intrinsic Linking Conjecture: Every graph on $n \geq 6$ vertices and $4n - 9$ or more edges is intrinsically linked.

Our approach was to prove a theorem using common neighbors whose result would imply Sachs' Intrinsic Linking Conjecture.

Theorem: If G is a graph with $n \geq 6$ vertices and $4n - 9$ or more edges, then G has a K_6 minor.

Robertson, Seymour and Thomas (1993) showed that a graph is intrinsically linked if and only if it has one of the seven Petersen graphs as a minor and K_6 is a Petersen graph. So if G has a K_6 minor then it is intrinsically linked. By proving it using this method we were able to come up with a much more specific result that for $n \geq 6$ vertices, $4n - 9$ or more edges guarantees a K_6 minor and not just intrinsic linking.

As a consequence of working through this proof there were several questions or conjectures that came about. The lower bound of $4n - 9$ edges that is used in Sachs' Intrinsic Linking Conjecture comes from taking a graph G that is nonplanar and adding

K_1. But what if you were to add K_2 to a graph G that is nonplanar? If G was planar with n – 2 vertices, you would then get an upper bound on the number of edges of 3(n – 2) – 6 + 2(n – 2) + 1 = 5n – 15. So a lower bound for $G + K_2$ to guarantee that G is nonplanar would be 5n – 14. We showed that if a graph has at least 4n – 9 edges then it has a K_6 minor. A natural continuance might be to see if a graph that has at least 5n – 14 edges necessarily has a K_7 minor. Conway and Gordon (1985) showed that a graph with a K_7 minor contains a nontrivial knot, so does at least 5n – 14 edges guarantee that a graph is intrinsically knotted (i.e. contains a nontrivial knot in every embedding)? Mader (1968) showed that if a graph with n vertices has at least 5n –14 edges then it has a K_7 minor, and hence is intrinsically knotted. Again Mader (1968) used a different technique, so it would be interesting to see if we can make an argument like the one presented in this thesis to prove the corresponding conjecture about intrinsic knotting.

Another question that came about from our work was whether we could come up with a generalization for complete minors. That is, what is the minimum number of edges for a graph with n vertices that would guarantee that the graph has a K_m minor? For a graph with n vertices, 3n – 5 edges guarantees a K_5 minor, 4n – 9 edges guarantees a K_6 minor, and 5n – 14 edges guarantees a K_7 minor. So the pattern we get is, if a graph with n vertices has $mn - \left[\binom{m+1}{2} - 1\right]$ edges then the graph has a K_{m+2} minor. In 1968, Mader showed that this was true for values up to $m = 5$, but doesn't work for $m \geq 6$. For example, even if a graph with n vertices has 6n – 20 edges, this does not guarantee that there will be a K_8 minor. Mader (1968) showed that $K_{2,2,2,2,2}$ (see Fig. 21) is a counterexample.

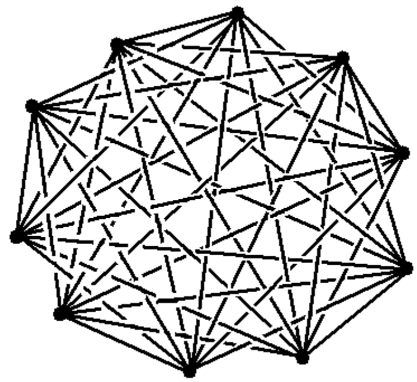


Fig. 21 An embedding of $K_{2,2,2,2,2}$

The graph has 10 vertices and 40 = 6(10) – 20 edges, but it does not have a K_8 minor. So because the pattern that we observe in the first few values of *m* does not continue, then the question becomes what is the least number of edges on *n* vertices that would guarantee a complete graph minor of K_8 and beyond.

Another similar question to the one that we looked at here is the idea of triple linking, that is, having a graph that contains three separate components that are linked together. Blain et al. (2005) showed that $G + K_5$ is intrinsically 3–linked if G is nonplanar. Now because this is similar to the way that we started this thesis with $G + K_1$ being intrinsically linked if G was nonplanar, then we should be able to estimate a value for the minimum number of edges that would guarantee triple linking. So if we have a graph G on $n – 5$ vertices that is planar then it can have at most $3(n – 5) – 6$ edges. If we add a K_5 to G then we get at most $3(n – 5) – 6 + 10 + 5(n – 5) = 8n – 36$ edges. Now if we add one edge to this graph for a total of $8n – 35$ edges then maybe this would be the number of edges that would guarantee triple linking. Mader (1968) already showed that such graphs do not necessarily have a K_{10} minor but they still might be triple linked. In fact, Blain et al. (2005) generalized this idea and proved that $G + K_{5m+1}$ is intrinsically $(m + 2)$–linked when G is nonplanar. Using the same argument as above we arrive at a lower bound of $(5m+4)n - \left[\binom{5m+5}{2} - 1\right]$ edges being sufficient to guarantee that a graph is (m + 2) –linked. This is not to say that this is the best bound possible, but it could be a good place to start.

In our original approach to proving Sachs' Intrinsic Linking Conjecture, we introduced the idea of common neighbors. We were unable to prove Conjecture 1 (Let *G* be a simple, connected graph such that every pair of adjacent vertices has at least four common neighbors. Then *G* has a K_6 minor.), but we have some verification that this approach might work. In the proof of our main theorem we assumed that *G* at least $4n - 9$ edges, and we quickly showed that every pair of adjacent vertices must have at least 4 common neighbors. However, the lower bound of $4n - 9$ edges should not be part of the proof if we want to only use the idea of common neighbors throughout the proof. We need to restrict our hypothesis to using just common neighbors. The problem is that when we start to look at minors or subgraphs of *G*, the number of common neighbors can change. With some further research we may find that we can shorten the proof of Sachs' Intrinsic Linking Conjecture using conjecture 1 and then possibly apply this idea to other areas in knot and graph theory.

In this thesis we restricted our graphs to simple graphs, but what if you were to restrict it even more? For example, we could just look at bipartite graphs such as $K_{2,2}$ to see if there is a smaller number of edges that would assure intrinsic linking. Maybe we need less than $4n - 9$ edges for a graph of *n* vertices. We could then look at tripartite graphs and so on for a possible generalization to *k*–partite graphs.

Graph theory has been used as a tool in the study of knot theory. Graph theory has made it a little easier to visualize knots and we have been able to apply rules and theorems from graph theory to knots. By knowing what kinds of graphs are linked or knotted, we are able to look for specific graphs (e.g. Petersen Graphs) to determine whether or not we have a link or a knot. Also, because graph theory has been studied for a much longer time, there is a wealth of knowledge from which to draw.

Graph theory has been an asset to the study of knots, but what about the other way around? We mentioned earlier how the Jones Polynomial from knot theory has helped with the study of statistical mechanics, but what does the study of knots have to offer graph theory? We may find, like so many other things in mathematics, that more applications will come later. It may be that the idea of intrinsic linking or knotting may help with solving problems of connectivity or shortest paths in networks. For a long time we have been looking for ways to apply other fields of mathematics to knots, and in the future we will probably find more ways to apply the techniques we use in distinguishing knots from one another to other fields of mathematics.

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