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Graphs that are minor minimal not Apex of connectivity $\kappa \leqslant 5$

Honors in the Major Thesis

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Nomenclature

ρ	A graph property
$E(G)$	Edge set of a graph G
G	A finite simple graph
G^+	A graph with an added edge
G^-	Graph with a removed edge
$G_1 < G_2$	Graph G_1 is a minor of G_2
$MMNA$	A graph G that is NA and also Apex when actions G/e and $G - e$ are applied to G
NA	A graph G that is Not Apex is $G - v$ is not planar
$V(G)$	Vertex set of a graph G

Abstract

We embark on an exploration of graphs which are closed under taking minors. That is to say, a graph G with a certain property p, also has that same property p after we take a minor. In particular, we consider the case for all graphs that are apex by considering the finite class of graphs that are minor minimal not apex. Recall that a graph, G is apex if $G - v$ is planar for some $v \in V(G)$. For the consideration of this report we consider only graphs that have vertex connectivity equalling to five. Specifically is the conjecture that there exist only two such graphs that are minor minimally not apex which are also with connectivity $\kappa = 5$, namely K_6 and the prism graph on 6 vertices plus two additional vertices connected to all but themselves, which is called the Jørgensen graph.

Chapter 1

Introduction

In 2004 Neil Robertson and Paul Seymour published the article "Graph Minors.XX. Wagner's conjecture, . . . ,in which they proved that for every infinite set of finite graphs, one of its members is isomorphic to a minor of another" [4]. Meaning, that for graphs closed under taking minors, one can always find two graphs such that one of them is a minor of the other. We delve into the basics of Graph Theory to refresh the essential definitions and theorems that have been referenced for the purpose of this research. I begun the researching this topic in the Summer of 2016, continued on into my senior year of my BSc with Professor Thomas Matman for the purpose of honors in the major. I endeavour to delineate effectively the nature of Graph Theory as it pertains to the Graph Minor Theorem.[2]

1.1 Basics

We begin this discussion on graphs with the following essentials to any discussion in Graph Theory. Generally any textbook on Graph Theory will cover these topics. Take *Graph Theory* by Reinhard Deistel, see [2], for a complete look at graph theory.

1.1.1 Definitions

We recall the essentail definition of a graph:

Definition 1: Graphs

A **graph** is a pair $G = (V, E)$ of sets, with $E \subset V^2$; that is the elements of E are 2 element subsets of V. The $v \in V$ are called **vertices** and the $e \in E$ are called **edges**. We use the notation $e := uv$. If $u = v$ then G contains **loops** and potentially **double edges** and is called a **multi-graph**. We restrict multi-graphs and only consider **simple** graphs containing no double edges or loops.

Figure 1.1 shows an example of a simple graphs

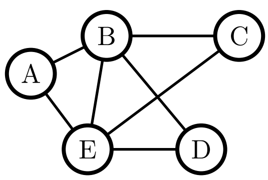


Figure 1.1: A simple graph on 5 vertices and 7 edges.

Order and size

For a graph G with n vertices and m edges, we say $|G| = n$ and $||G|| = m$, and $|G|$, $||G||$ is called the **size** and **order** respectively.

Incidence, Adjacency, Neighbours, Degree

For any edge there are 2 **ends**, say u and v. We say that uv is incident upon u or v. Also, we say that u and v are **adjacent**. Suppose that u is adjacent to k vertices, then the **neighbourhood** of v is the list of all k vertices and we denote this list by $N(v)$, and its length or size by $|N(v)|$. The **degree** of $v = |N(v)$. The graph G where all vertices are adjacent to all other vertices in G is said to be **complete** and is denoted as K_n. We say the *complete graphs on n vertices.*

Subgraphs

Let H be a **subgraph** of G, where H and G are both graphs and we notate as $H \subset G$. Then $V(H) \subset V(G)$ and $E(H) \subset E(G)$. If $V(H) = V(G)$, then H is a **spanning** subgraph. If our subgraph $V(H)$ is an arbitrary subset of $V(G)$, and $E(H)$ consists of all edges of G whose ends are all in $V(H)$, then H is an **induced** subgraph of G.

1.1.2 Operations of Graphs

Let G be a graph, then a **graph operation** is defined in the following boxes.

Definition 2: Vertices and Edges

Vertices

For a graph G, we can add or subtract vertices.

- $G - v$, is the graph G without v and all edges incident upon v.
- $G + v$, is the new graph $G + v$ with a new vertex v added to the old graph G, that may or may not have added edges.

Edges

For a graph G, we can add or subtract edges.

- $G - e$, is the graph G with an edge removed from the existing graph.
- $G + e$, is the existing graph G, where two non-adjacent vertices are now adjacent after $G + e$.

Definition 3: Edge Contractions

Let x and y be two adjacent vertices in a graph G, then G/xy is the graph formed from smashing x and y into a new vertex, z. We note that $|G/xy| \leqslant |G| = n - 1$. Then this is called an **edge contraction**

Figure 1.2 shows $K_5/v \cong K_4$, through an edge contraction.

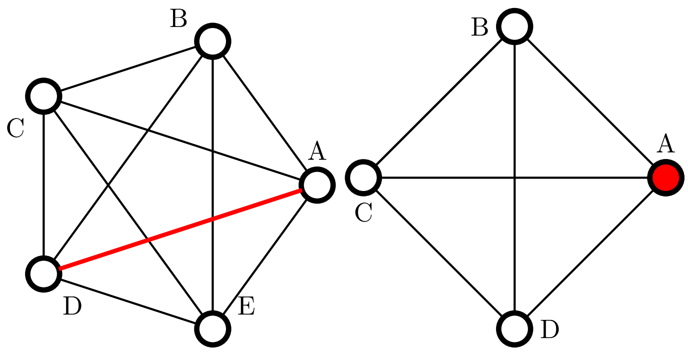


Figure 1.2: Edge contraction $K_5/e \cong K_4$

Definition 4: Vertex Splits

Let G be a graph and let z be a vertex on n neighbors. A **vertex split** splits z into two new vertices x and y. We write $G \star v(a, b)$, where $a = |N(x)| - 1$ and $b = |N(y)| - 1$.

Figure 1.3 shows a 4 different vertex splits. The top left is written as $G \star A(0, 3)$. The bottom right is written as $G \star A(3, 3)$, which is also isomporphic to K_5. Not all vertex splits will be the inverse operation of a given edge contraction.

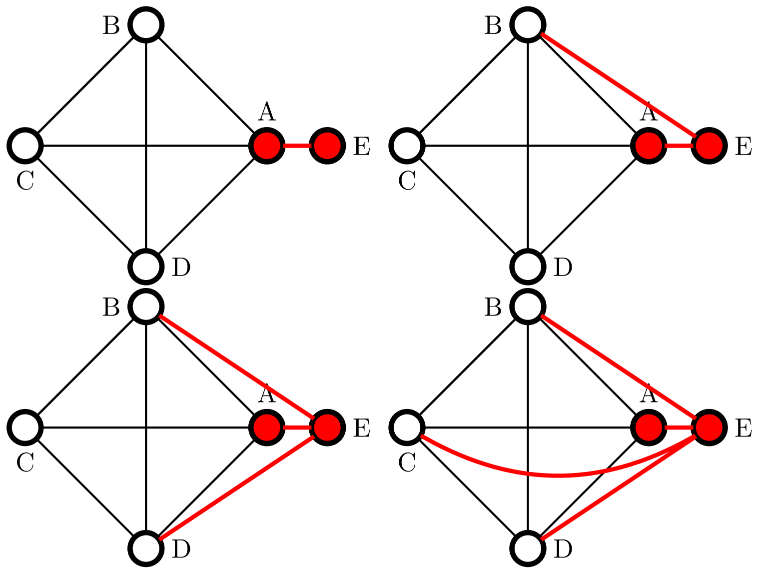


Figure 1.3: Vertex split options of K_4

Definition 5: Connectivity

Let G be a connected graph, U be a subset of vertices in G, and $|U| < \kappa$. G is $\kappa-$connected if $G - \kappa$ is connected

Connectivity

A graph is connectivity $\kappa = k$, when κ is largest integer for which $G - k$ is $\kappa-$connected.

1.2 Graph Minor Theorem

1.2.1 Minors

Definition 6: Graph Minor

Let a graph H be a **minor** of another graph G. Then at least one of the following is true:

1. $G - v \cong H$
2. $G - 3 \cong H$ item $G/e \cong H$.

We say that H is a minor of G and write $H < G$. Let only one of the above occur, then we say that H is a **simple minor** of G.

Theorem 1: Graph Minor Theorem

For every infinite set of finite graphs, there exists $G_i < G_j$ for $i \neq j$. In essence one of its members is isomorphic to a minor of another. For a detailed proof see [4].

The Graph Minor Theorem is one of the deepest results in Graph Theory today. The theorem is existential and not constructive. Implications of this theorem are far reaching and is the foundation of this thesis.

Definition 7: Graph Property

Recall that a **graph property** is a class of graphs that is closed under isomorphism, one that contains with every graph G also the graphs isomorphic to G. See [2]. Then, a graph G is either P or not P.

Definition 8: Minor Closed

Let G be a graph with P as a graph property, then G is **minor closed** when all minors of G also have property P.

Definition 9: Minor Minimality

Let G be a graph is property P, then G is minor minimal when a minor is not P.

What is more, it suffices to check simple minors only. Moreover, the size of the class of $MMNP$ graphs is finite.

Corollary 1: Finite Obstructions

The set of MMP graphs is finite.

Proof: Finite Obstructions

Let $\mathcal{F} = G_1, G_2, ...$ be the set of MMP graphs. By the Graph Minor Theorem (**GMT**) there exists $G_i < G_j$. This contradicts that $\mathcal{F}$ is the set of all MMP, G_j can have no proper minors that have P.

If P is minor closed we can write

$$Obs(P) := \{G - \text{Minor Minimal for not P } (\mathbf{MMNP})\}. \tag{1.1}$$

Corollary 2: Complete Class Characterization

Let P be minor closed, then

$$G_P \Leftrightarrow G \cap Obs(P) = \emptyset, \tag{1.2}$$

Where G_P is the graph G with property P.

That is to say that is P is minor closed then $Obs(P)$ completely characterizes P. As a further matter, if P is not minor closed we do not have the if and only if characterization. However, if G has P, G has a minor in the finite list of MMP. So this leads to prove G does not have P. Essentially we show that it has no MMP minor. The problem is, even though G is not P, it may have a minor that is P. If $\mathcal{F}$ is minor closed and there exists a graph G that has $\mathcal{F}$, then $K_1 < G$ and K_1 has $\mathcal{F}$. If there exists G' that has not $\mathcal{F}$, then $K_1 < G'$. So not $\mathcal{F}$ is **not** minor closed.

Definition 10: Minor Minimal not P

Let G be a class of graphs that is minor closed with property P, then G is minor minimal not P - or $MMNP$ - if G is not P and every minor is P.

We will see that is definition leads into our study of Apex graphs.

1.2.2 The Family of Apex Graphs

Definition 11: Apex Graph Property

Let $\mathcal{F} := \{G \subset \mathcal{F} \,|\, \exists v \in V(G) \ : \ G - v \text{ is planar.}\}$, then $\mathcal{F}$ is the class of **Apex** graphs, which is minor closed.

Lemma 1: Planar is Apex

All planar graphs are Apex

Proof: Planar is Apex

Let $u, v \in V(G)$ for a graph G. Suppose G is apex, then $G - v$ is planar, and $G - u, v$ is also planar. Then planar graphs are also apex.

Definition 12: Not Apex

A graphs, G, is not apex, if for all $v \in V(G)$, $G - v$ is not planar.

Theorem 2: Wagner and Kuratowski

A graph G is planar if and only if it does not contain a K_5 or $K_{3,3}$ minor. For a detailed proof see [2].

1.2.3 Minor Minimal Not Apex

Definition 13: Minor Minimal Not Apex

Let G be not apex, and let every proper minor of G be apex, then G is $MMNA$.

In fact since planarity is a minor closed property, both K_5 and $K_{3,3}$ completely characterize planar graphs. In like manner the complete list of $MMNA$ graphs characterizes all Apex graphs.

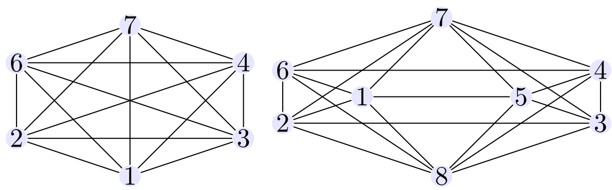
1.3 Results

In seeking the list $MMNA$ graphs, a number of results have emerged. In order to manage this, graphs have been broken down into connectivity. In essence connectivity, $\kappa = 1, 2, 3, 4, 5, ...$ There are no $MMNA$ $\kappa = 1$ or $\kappa > 5$. [3].

1.3.1 K_6 and Jørgensen graphs

We look at K_6 and the Jørgensen graphs

Figure 1.4: K_6 and J



Our main conjecture is the purpose of this thesis.

Conjecture 1: K_6 and J $MMNA$

The two $MMNA$ graphs with connectivity $\kappa = 5$ are K_6 and the Jørgensen graph, called J.

Moreover, a weaker conjecture is studied as well.

Conjecture 2: G $MMNA$ $\kappa = 5$

et G be an $MMNA$ graph with $\kappa = 5 \Rightarrow K_6^- < G$

Strategizing to a proof has generated the following path.

Strategy: Moving to a solution

We look at two main approaches to completing the list:

1. We note that $K_6^- < K_6$ and $K_6^- < J$
   - G is $MMNA$, and connectivity $\kappa = 5$ with K_6^- being a minor, then G is one the conjectured two.
   - This has been proved for $|G| \leqslant 13$.
2. It is known that K_4 is $3-$connected, Tutte's proved that K_4 can be used to generate all other $3-$connected graphs. For any of these graphs say, G_i, take $G + a, b$ with at least five edge additions, then $G + a, b$ is tested for $MMNA$ with connectivity $\kappa = 5$. This method clearly generates a large number of graphs testable for $MMNA$, connectivity $\kappa = 5$. Either $G_i + a, b$ is K_6 or J, or we have found a new $MMNA$ graphs with connectivity $\kappa = 5$.

Chapter 2

What do we know about $MMNA$ graphs

2.1 Introduction

Currently a complete characterization of Apex graphs is incomplete. Investigations are under way to classify the list of obstructions. We look at what is known currently on the matter of $MMNA$ graphs.

2.1.1 Minor Minimal Intrinsically Knotted

Definition 14: Linkless Embedding

A graph G, with an embedding in $\mathbb{R}^3$ is called a **linkless embedding**, if there are no linked cycles.

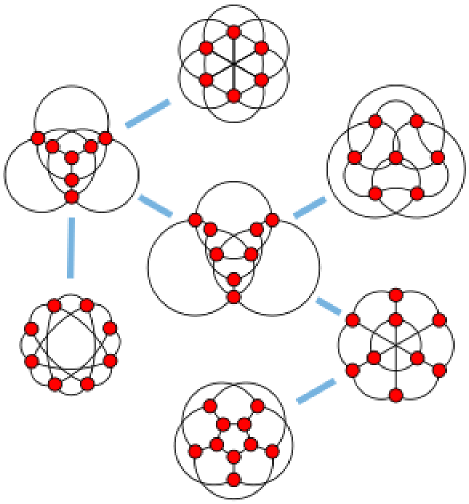
The property, linkless embedding, is minor closed and hence the complete list of obstructions completely characterizes linkless embedded graphs.

Theorem 3: Minor Minimal Intrinsically Linked - $MMIL$

The family of graphs known as the Petersen Graphs, are the complete list of forbidden graphs for linkless embeddings. Proved in 1995 by [5].

We note the Petersen family

Figure 2.1: The Petersen Family - creative commons



The 7 graphs of the Petersen family form the complete list of $MMIL$. The set of all intrinsically linked graphs are not apex. Moreover the $MMIL$ are $MMNA$. A graph G is outer-planer if it is planar and all vertices are incident upon a common face. Further more a graph is apex outer-planar if there exist $v \in V(G)$ such that $G - v$ is apex outer-planar. In turns out there are 57 such graphs that are minor minimal not apex outerplanar.

2.2 $MMNA$ Graphs

Recall that G is $MMNA$, if G is not apex and every proper minor of G is apex.

2.2.1 Disconnected $MMNA$ graphs

Theorem 4: Disconnected $MMNA$

The three disconnected $MMNA$ graphs are

1. $K_5 \sqcup K_5$
2. $K_{3,3} \sqcup K_{3,3}$
3. $K_5 \sqcup K_{3,3}$

2.2.2 Connectivity of $MMNA$ graphs

As mentioned, in order to manage the research, the study of $MMNA$ graphs is broken into different connectivities. Meaning we look wat $\kappa = 2, 3, 4, 5$ and exclude $\kappa = 1, \geqslant 6$.

Theorem 5: No $MMNA$ with connectivity $\kappa = 1$

Let G be a graphs with connectivity $\kappa = 1$, then G is not MMNA.[3]

We note that for $MMNA$ graphs of connectivity $\kappa = 2$, and 3 there are over 100 known graphs for each. For $\kappa = 4$ there are at least 6, and for $\kappa = 5$ it is conjectured only 2. A computer search found 263 known graphs. Professor André Kézdy and his team have reportedly discovered 396 such $MMNA$ graphs. It is believed then, there are approximately 400 $MMNA$ graphs.

Lemma 2: $\kappa \leqslant \delta$

Let G be a graph, then $\kappa \leqslant \delta$

Proof

Let $\delta(G) = d$ for a graph G.Then there exist $v \in V$ such that $|N(v)| = d$. Then $G - N(v)$ disconected v from G and $\kappa \leqslant d$, and $\kappa \leqslant \delta$.

Theorem 6: $\delta(G_{MMNA}) \leqslant 5$

If G is $MMNA$, then $\delta(G) \leqslant 5$ [3]

Theorem 7: $\kappa(G_{MMNA}) \leqslant 5$

If G is $MMNA$, then $\kappa(G) \leqslant 5$ [3]

Theorem 8: $\delta(G_{MMNA}) = 5$

If G is $MMNA$ with $\kappa(G) = 5$, then $\delta(G) = 5$.

Proof

Theorem 6 tells us that G $MMNA$ implies $\delta \leqslant 5$. We looking for graphs with $\kappa = 5$, hence $5 = \kappa \leqslant \delta \leqslant 5$.

Theorem 9: Bollobás: Let G have $\delta \geqslant 5$

If $\delta(G) \geqslant 5$, then either $K_6^- < G$ or $I_{12} < G$ [1]

2.3 Connectivity 5 and $MMNA$ K_6 and J

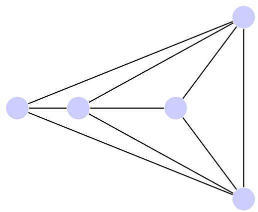
As conjectured, the only two $MMNA$ graphs of connectivity $\kappa = 5$ are K_6 and J.

Lemma 3: K_6 is $MMNA$

et G be the complete graph on 6 vertices, called K_6, then K_6 is $MMNA$.

Proof: K_6 **is** $MMNA$

elect $v \in K_6$, due to rotational symmetry, any one vertex will suffice to show that K_6 is not apex. Let, $K_6 - v$, this is isomorphic to K_5, which is not planar. Hence K_6 is not apex. Next, select any one $e \in K_6$, then $K_6/e \cong K_5$, which is apex. Next take $K_6 - e$, and delete a vertex with $deg(v) = 5$. The remaining graphs is

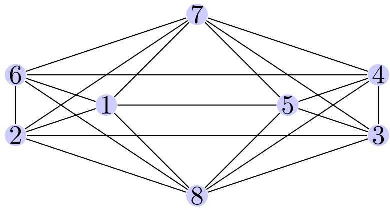


Hence K_6 is $MMNA$.

Lemma 4: Jørgensen graph is $MMNA$

The Jørgensen graph is $MMNA$

Proof



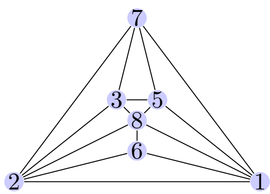
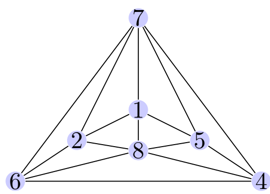
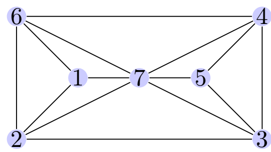
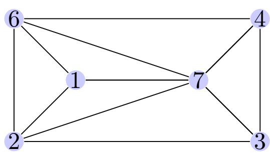
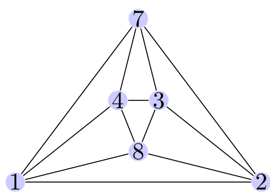
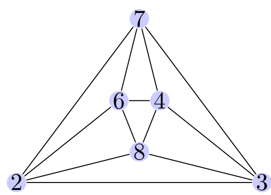
We observe that there are two sets of vertices,

$$V_1 = \{7, 8\} \tag{2.1}$$
$$V_2 = \{1, 2, 3, 4, 5, 6\} \tag{2.2}$$

There are also three edges sets.

$$E_1 = \{(1, 7), (2, 7), (3, 7), (4, 7), (5, 7), (6, 7), (1, 8), (2, 8), (3, 8), (4, 8), (5, 8), (6, 8)\} \tag{2.3}$$
$$E_2 = \{(1, 2), (2, 6), (1, 6), (3, 4), (4, 5), (5, 3)\} \tag{2.4}$$
$$E_3 = \{(1, 5), (4, 6), (2, 3)\} \tag{2.5}$$
$$\tag{2.6}$$

Consider that $J/_{(1,5),(3,4)} - 7 \cong K_5$ and $J/_{(1,5)} - 3 \cong K_{3,3}$. With respect to V_1 & V_2, J is **NA**. It turns out that these are the 6 planar graphs proving J is $MMNA$.

<table>
<tr><th>MM - J</th><th>E_1</th><th>E_2</th><th>E_3</th></tr>
<tr><td>$J - e - v$</td><td>
$J - (7, 6) - 4$</td><td>
$J - (1, 6) - 3$</td><td>
$J - (1, 5) - 8$</td></tr>
<tr><td>$J/_e - v$</td><td>
$J/_{(7,5)} - 8$</td><td>
$J/_{(1,6)} - 6$</td><td>
$J/_{(1,5)} - 5$</td></tr>
</table>

2.4 3−connected constructions

We show that it is possible to construct inductively 3−connected graphs from K_4.

Lemma 5: 3−connected Still

All 3−connected graphs $G \neq K_4$ has an edge e such that G/e is again 3−connected.

Theorem 10: 3−connected Criteria

A graph G is 3−connected if and only if there is a sequence $G_0, ..., G_n$ such that: $G_0 = K_4$, $G_n = G$, and G_{i+1} has an edge xy such that $\deg(x), \deg(y) \geqslant 3$ and $G_{i+1}/xy = G_i$ for $i < n$. What is more, all graphs in the sequence are all 3−connected.

Hence, all 3−connected graphs can be constructed inductively, given a 3−connected graph to start from. Simply pick any vertex v in G a 3−connected graph and split it into two new vertices, v' and v". Then join these to all former neighbors of v, each to a least 2 other vertices. Let a graph of the form $C_n \star K_1$, be called a wheel. Thus K_4 is the smallest wheel.

2.5 Content to Come

It is shown in chapters 3 and 4, the strategies and work that has already been done. Chapter 3 expresses the main conjecture, the auxiliary conjecture and the strategies in place to weed out a proof. In chapter 4, it is seen that 3−connected graphs are used to determine $MMNA$ by brute force, with the aim to determine $MMNA$ of connectivity 5.

Chapter 3

The root of connectivity $\kappa = 5$ $MMNA$ is K_6^-

3.1 Introduction

The main conjecture of this thesis is that the complete graph on 6 vertices and the Jørgensen graph are the only two $MMNA$ graphs with connectivity 5. An axillary conjecture that is also conjecture, that is also explored, is G, being NA, $\kappa = 5$, implies that $K_6 - e$ is a minor of G.

3.1.1 Properties of $MMNA$

We recall the definition of $MMNA$, then G $MMNA$ is not apex and every proper minor of G is apex.

Definition 15: $n-$apex

Let G be a graph, $U \subset V(G)$, $|U| = n$, $G - n$ is planar, then G is $n-$apex.

Theorem 11: $MMNA$ graphs are $2-$Apex

Let G be an $MMNA$ graph, then G is $2-$Apex.

Proof

Suppose $G - v$, $v \in V$. Since G $MMNA$, G is NA and every proper minor is apex, then there exists $u \in V(G - v)$ where $G - u, v$ is planar.

Notation

Let $G - e$, then we write G^-.

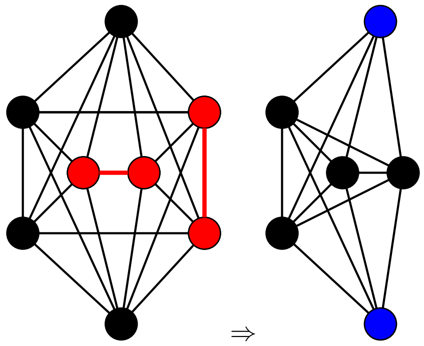
3.1.2 K_6^-

Lemma 6: Minor of K_6 and J

Let K_6 and J, then K_6^- is a minor of them both.

Proof

Clearly $K_6^- < K_6$, take any edge, e and $K_6^- \cong K_6 - e$, done. To see the J case. Note the subgraph



There is a K_4 minor, that is connected to the two blue vertices, it is easy to check the this is isomorphic to K_6^-.

Recall Theorem 9, that if G has $\delta \geqslant 5$, then $K_6^- < G$ or $I_{12} < G$. What though if G is not-planar, or not apex, can we restrict Bollobás further?

3.2 Bollobás

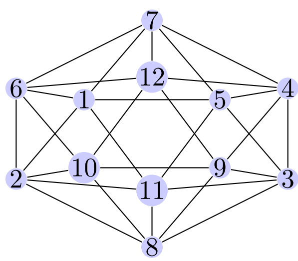
A graph G, with $\delta \geqslant 5$, has either a K_6^- minor or I_{12} minor. We know that is G is connectivity 5 and $MMNA$ and K_6^- minor, then G is either K_6 or J.So finding a K_6^- minor is of little consequence at this time. We are instead concerned if, $I_{12} < G$ and G is NA. Does G, then have K_6^- as a minor? In fact, we believe this is the case.

Conjecture 3: G NA Connectivity 5

Let G be NA, connectivity 5, then $K_6^- < G$

Hence, we will consider ways in which we have both an I_{12} minor and a K_6^- minor. Recall the Icosahedral graph

Figure 3.1: The Icosahedral Graph, I_{12}



Notation

Let $G + e$, then we write G^+. Furthermore, if $G + 2e$, then we write G^{++}.

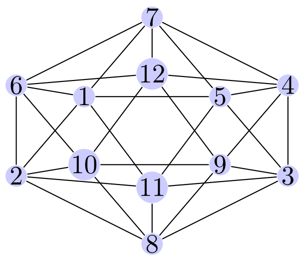
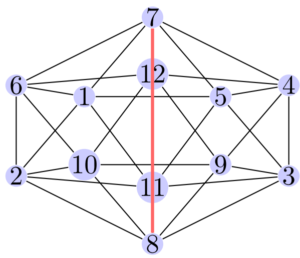
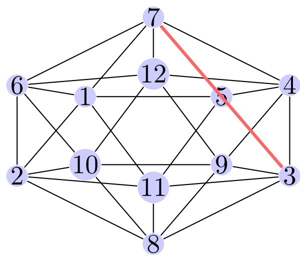
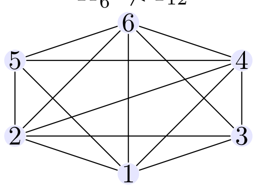
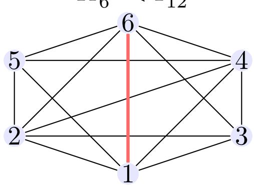
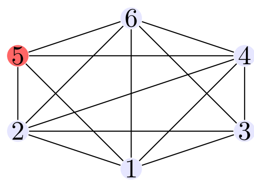
Lemma 7: $K_6^- < I_{12}^+$

Let K_6^-. And suppose we have I_{12}^+, then $K_6^- < I_{12}^+$.

Proof

To see a proof, observe the table below. At present if $G = I_{12}$, there is no K_6^- as I_{12} is planar and K_6^- is apex. However, consider that $I_{12} + e$, written as I_{12}^e is apex, due to symmetry there are only two possible edge additions. In fact in both cases, K_6^- is a minor.

Figure 3.2: $K_6^- < I_{12}^+$

Case 0	**Case** 2	**Case** 3
 I_{12}	 I_{12}^+ Antipode	 I_{12}^+ Not Antipode
$K_6^- \not< I_{12}$ 	$K_6^- < I_{12}^+$  $I_{12}^+ : 8 \to 6, 7 \cong 5, 10, 11 \to 2$ $9 \to 2, 6 \to 3, 0, 5 \to 1$	 $I_{12}^+ : 12, 11, 5 \to 4, 11 \cong 4$ $10 \cong 3, 7 \cong 1, 4, 6, 8 \cong 5$

We can take this further, suppose the G is not planar and has I_{12} as a minor.

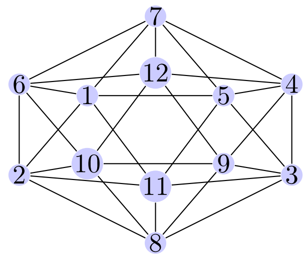
Lemma 8: NP from I_{12}

Let G be a non planar graph with $\delta(G) \geqslant 5$, generated from I_{12} by a single vertex split, then $K_6^- < G$.

Proof: Regarding Lemma 4

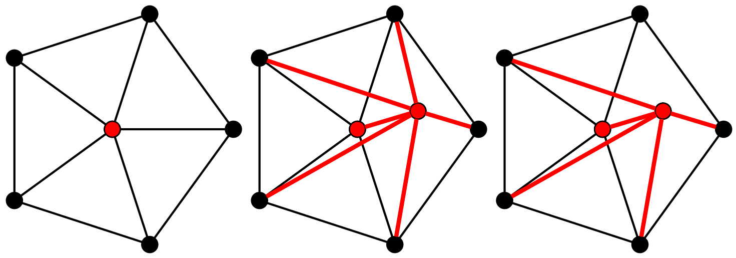
Suppose for simplicity we select vertex 7 and operate a single vertex split.

Figure 3.3: I_{12}



We will consider a top view of only the subgraph W_5.

Figure 3.4: I_{12} top



Let the split vertex be the red vertex. Call each v_1 and v_2, suppose $deg(v_1) \leqslant deg(v_2)$ and suppose that $N(v_1) = \{a, b, c, d\}$ and $N(v_2) = \{a, b, c, d, e\}$. For a, b, c, d, e being arbitrary letters denoting the 5 cycle.

Case 1: $deg(v_2) = 6$

We contract the edge v_2e, then $N(v_1) = \{a, b, c, d, e, v_2\}$ and we have a new graph isomorphic to I_{12}^+, which we know $K_6^- < I_{12}^+$.

Case 2: $deg(v_2) = deg(v_2) = 5$

Then all of the vertices a, b, c, d, e will be connected to anyone of the vertices v_1, v_2. Say $N(v_1) = \{a, b, c, d, v_1\}$ and $N(v_2) = \{a, b, c, e, v_1\}$. Contract, say edge $= v_2e$, then $N(v_1) = \{a, b, c, d, v_2\}$ and say v_2c is an edge of previously non adjacent vertices and we have a graph isomorphic to I_{12}^+, which we know $K_6^- < I_{12}^+$.

It is important to realise that if a graph H and a graph G have the relation $H < G$, there there exist some order such that

$$H = G_0 < G_1 < \cdots < G_n = G \tag{3.1}$$

Where each G_i is obtained by either operations to obtain minors:

- Vertex deletion,
- Edge deletion,

- Edge contraction,

or operations to generate a new graphs with a specific minor:

- Vertex addition,
- Edge addition,
- Vertex split.

In either case, the operations commute, so we can take the operations that make minor identification easiest first, and we would know that all subsequent graphs would have the original graph as a minor as well.

Theorem 12: $|13|$ or Less

Non planar graphs of size 13 or less and $\delta \geqslant 5$ have K_6^- as a minor.

Proof

Firstly, we consider that if G has $\delta(G) \geqslant 5$ and $|G| \leqslant 11$, then automatically we K_6^- as a minor, so we consider graphs from $|G| = 12$ and $|G| = 13$.

$|G| = 12$

By **Theorem 12** and excluding K_6^- we have $I_{12} < G$ By **Lemma 4**, we know that if $I_{12} = G_0 < G_1 <, ..., < G_n = G$ is non planar and $|I_{12}| = |G|$, then we are looking for edge additions that will make I_{12} not planar. In this case, the icosahedral graph is maximally planar and any edge addition will give the desired results. by a single edge addition, then rearrange operations and do those first. We know from Lemme 3, that $K_6^- < G$.

$|G| = 13$

We discard the case where K_6^- is already a minor of G, but look at $I_{12} < G$. As before we know that order of minors operations commutes, so we put the most logical first in the list. In this case, $|I_12| + 1 = 13 = |G|$. We know that there must be a vertex split in I_{12} with $\delta \geqslant 5$. **Draw and test** I_{12}.

Conjecture 4: G_{NA} and $\delta \geqslant 5$ implies $K_6^- < G$

Let G be a not apex graph with $\delta \geqslant 5$, then $K_6^- < G$

Strategy: Ways to a Proof

1. Generate graphs with $\delta \geqslant 5$ to test K_6^- minors.
2. Observe patterns in graphs with vertices greater then 13 and with $\delta \geqslant 5$ and test for $K_6^- < G$

Conjecture 5: The two $MMNA$ graphs of $\kappa = 5$

K_6 and the Jørgensen graphs are the only $MMNA$ graphs of connectivity $\kappa = 5$.

We end this chapter on the main conjecture of the thesis. In truth every item in this chapter is a lead up to this problem. Further to this discussion, in which we see the $MMNNA$ G with connectivity 5, seem to have a K_6^- minor. In a sense, shrinking down $MMNA$ to a mutual minor. However, we look forward to chapter 4 in which we, to use the same metaphor, expand K_4 to being a candidate for $MMNA$ through various operations to be discussed.

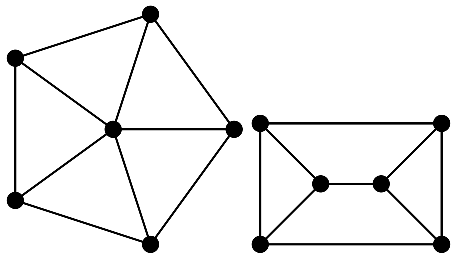
Chapter 4

Constructing $MMNA$s

4.1 Graphs that are $3-$connected

Recall that a graph G is $\kappa-$connected, when there exists $U \subset V(G)$, with $|U| = \kappa$, and $G - |U|$ is still connected. When $|U| = 3$, G is a $3-$connected graph. Recall as well that the **Wheel** graph, notated W_n is comprise of a cycle C_n and a singleton vertex, in which every vertex in C_n has degree two and then the singleton vertex is connected to all $v \in C_n$. Hence for a wheel graph, $\delta = 3$ and for the sigleton vertex, say a, the $N(a) = C_n$ and $\deg(a) = n$. In like manner if we split a, and replace with say v_1 and v_2, the planar graph the results is called the **Double Wheel** graph. We notate as DW_n. Where $N(v_1 \cap v_2) = C_n$, and $v_1 \cap v_2 = v_1v_2$, in that they are adjacent.

Figure 4.1: W_5 and DW_4



This chapter will discuss the generative process for constructing $3-$connected graphs G from K_4. We, then add two vertices and sufficient edges to ensure $\delta(G) \geqslant 5$. Tutte's theorem provides the basis for this graph constructor.

4.2 Tutte

The graph K_4 is a $3-$connected graph. Tutte proved the following.

Theorem 13: Tutte 1961

A graph is $3-$connected if and only if there exists a sequence $G_0, ..., G_n$ of graphs with the following properties:

1. $G_0 = K_4$;
2. G_{i+1} has an edge xy with $d(x), d(y) \geqslant 3$ and $G_i = G_{i+1}/xy$ for every $i < n$.

Moreover the graphs in any such sequence are all $3-$connected.For a proof see [2].

We recall from chapter two, that we can construct 3−connected graphs from K_4, by splitting a vertex, from say v to v' and v''. Then we attach v' and v'' to at least two vertices that were connected to v previously.

Algorithm 1: Generating 3−connected Graphs

3−connected generator through vertex splits

Suppose that $K_4 = G_0 < G_1 < G_2 < \cdots < G_i < \cdots$, with $i \in [0, \infty)$. Let

1. Let $K_4 = G_0$
2. Spit $G \star v(a, b)$ into v' and v''
3. Ensure that $|a + 1| \geqslant 3$ and $|b + 1| \geqslant 3$.
4. Ensure that v' and v'' adjacent to at least 3 edges, of which all were adjacent to v prior to the split.

Select graph G_i at some desired point and G_i will be 3−connected and what is more all such graphs in $\{K_4, G_1, ..., G_i\}$ are also 3−connected.

This algorithm will generate all 3−connected graphs. For our purposes we have decided it easiest to test graphs based on the size of graph G. At the time of writing, we have test up to 6 vertices for the 3−connected graph. Following we look then at the algorithm for generating a 5−connected graph from results obtained in Algorithm 1.

Algorithm 2: Test Algorithm

Not Apex Graph Generator Suppose we have $K_4 = G_0 < G_1 < G_2 < \cdots < G_i < \cdots$, with $i \in [0, \infty)$, as in Algorithm 1. Let $H := \{G_i : |G_i| = n\}$.

1. Add 2 new vertices, $G_i + 2v$, and edges, $G_i + e_1, ..., e_n$ until $\delta(G_i) \geqslant 5$
2. Test for $MMNA$
   (a) If G_i is $MMNA$
      i. Either G_i is K_6 or J
      ii. Else G_i is a new $MMNA$, or is not $MMNA$.
   (b) If G_i is not $MMNA$
      i. Add edges to G_i, and test for $MMNA$
      ii. Discard changes made to G_i and go back to Algorithm 1.
3. Stop when all $MMNA$ have been discovered

With these two algorithms in place, we have found some interesting results. As well as implemented the `plantri` program to find all planar graphs of desired size with connectivity being at least 3. From there, we simply implement Algorithm 2.

4.3 Creating candidates for $MMNA$ tests

There is no searchable computer algorithm that directly generates all 5−connected graphs, at least none that are fast. There is however known algorithms for generating 3−connected planar graphs in very fast time. The `plantri` computer algorithm will generate all planar graphs with certain characteristics. In our case we can generate all graphs on say n vertices with vertex connectivity equalling k. It was shown already that $MMNA$ graphs are 2−apex. Meaning that if a graph G is

$MMNA$ than there exists two vertices say, u, v such that $G - u, v$ is planar. Hence constructing all 5−connected graphs for $MMNA$ testing is straightforward.

Lemma 9: Construcint K_6 from 3−connected G

The $MMNA$ graph K_6 with connectivity $\kappa = 5$ is generated from K_4, a 3−connected graph and two additional vertices.

Proof

Obviously $K_4 < K_6$, take $K_6 - u, v$, due to the symmetry of K_6 any two vertices will suffice, and what is left is $K_6 - u, v \cong K_4$, which is a 3−connected graphs. Conversely take K_4 and two singleton graphs E_1 and E_1 then connect all vertices in K_4 to both vertices of the E_1 graphs, and K_6 is formed.

Lemma 10: $J = DW_4 + E_1 + E_1$

The $MMNA$ graph, Jørgensen, J is generated from the 3−connected graph, the prism graph on 6 certices, $DW_4 + E_1 + E_1$, where $E_1 \cap E_1 = \emptyset$.

Proof

Isolate the two vertices of $d(v) = d(u) = 6$ and $J - u, v \cong DW_4$. We know that $K_4 < DW_4$, and by Theorem 13 DW_4 a 3−connected graph. Moreover, take $DW_4 + E_1 + E_1$ with the same proviso that $E_1 \cap E_1 = \emptyset$ and we have a graph isomorphic to J, which is $MMNA$ and connectivity $\kappa = 5$.

Tutte's theorem tells us that any K_4 is a universal minor of all 3−connected graphs. We see that K_4 acts as a minor for K_6 and J and that both of these $MMNA$ graphs, can be constructed from K_4.

4.3.1 3−connected to 5−connected

Clearly the case holds in Lemma 5 and 6, what happens though in the general sense for generating a 5−connected graph from a 3−connected graph? In general we refer to the fact that for any graph G, then $\kappa(G) \leqslant \delta(G)$. As long as we are careful and ensure that all of our graphs generated from Algorithm 1 and then 2 we can test for connectivity of appropriate size and therefore for $MMNA$.

4.4 Results of graph construction upto 8 vertices.

In each of the following subsections when we refer to $|G| = n$, this is the total size of the planar graph plus two addition vertices, and hence the original 3−connected graph that was generated is actually $n - 2$ in size.

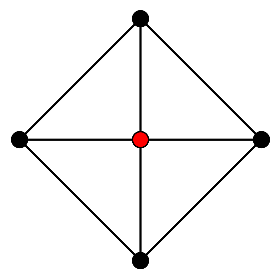
4.4.1 $|G| = 6$

This is the an $MMNA$ graph of connectivity $\kappa = 5$ and on 6 vetices, hence $|G - u, v| = 4$ and is 3−connected, in particular K_4. By Lemma 5, we know this is $G \cong K_6$. Hence, the $MMNA$ graph on 6 vertices is K_6. There is only one graph on 4 vertices that is 3−connected, that is K_4.

4.4.2 $|G| = 7$

In the case of G having 7 vertices, we have the following planar graph on 5 vertices Let vertex a,

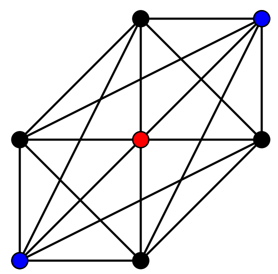
Figure 4.2: W_4



be the red vertex, and let the black vertices be in the cycle, C_4. And suppose we add two vertices, in blue, say u and v. Each vertex in blue connected to a vertex in black. Now $\deg(a) = \deg(u) = \deg(v) = 4$. We require $\delta \geqslant 5$, cases that satisfy this are:

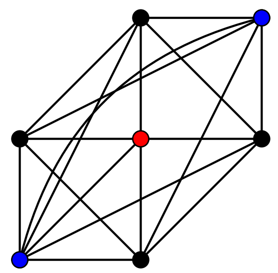
Case **a** Let $u \cap v = \emptyset$, and both blue vertices connected to the red vertex. Now $\delta = 5$ and this graph is apex.

Figure 4.3: Case **a**



Case **b** Let the red vertex be connected to only one blue vertex and both blue vertices, be connected to each other, then this graph is apex.

Figure 4.4: Case **b**



Case **c** Finally, let the two blue vertices be adjacent, and let as well the red vertex be adjacent to both blue vertices, then K_6 is a minor.

Hence, there is no $MMNA$ graph, connectivity 5, on 7 vertices. Interestingly, when the planar graph is a wheel on n vertices, we generalise the case of $|G| = 7$.

Lemma 11: W_n graph for $MMNA$

Let G be 2−apex, $\delta \geqslant 5$,with planar sub graph isomorphic to a W_n, then $K_6 < G$ or G is apex.

Proof

Let $|W_n| = n$, we know that the rim, or cycle of W_n will have each vertex with deg $= 5$. Hence, we are only concerned with the arrangement of the axle vertex and the two additional vertices. As seen in the above subsection, we have three options. Two options have G as apex and one that has K_6 as a minor.

Hence, regardless of size G as in the case of Lemma 11, will be apex or have K_6 minor.

4.4.3 $|G| = 8$

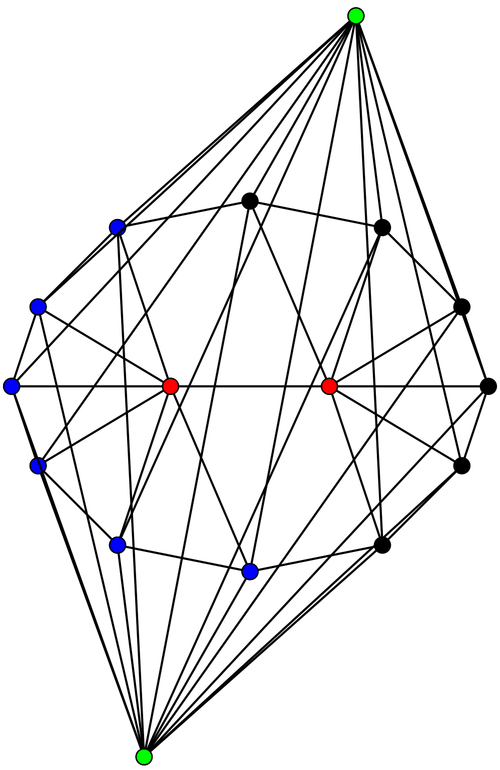
The $MMNA$ candidate generated from all 3−connected planar graphs on 6 vertices, have either a wheel graph, W_5 for a DW_4 graph as a planar minor. In the case of W_5, we see in Lemma 11 that K_6 is a minor of the graph is apex. In the cases containing a subgraph of DW_4, we have two options. Option 1, the planar 3−connected graph is only the DW_4 graph and option 2, the planar 3−connected graph is the DW_4 graph plus a variety of edges. We prove the case of option 1 in the generalized from of n vertices of DW_n.

Theorem 14: DW_n $MMNA$

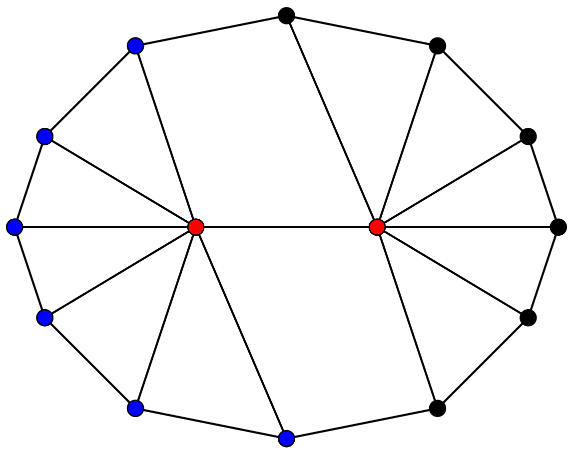
Let G be not apex, $\delta \geqslant 5$, and $G - a, b$ for $a, b \in V(G)$ is DW_n, the G is not $MMNA$.

Proof

Suppose G is of the form

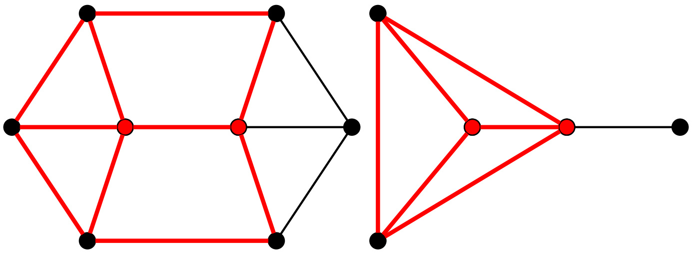


Where $|G| = n + 4$, n is the number of vertices on the rim of the DW_n graph, then 4 more vertices for a, b, c_1, c_2. Let DW_n have an embedding similar to the following

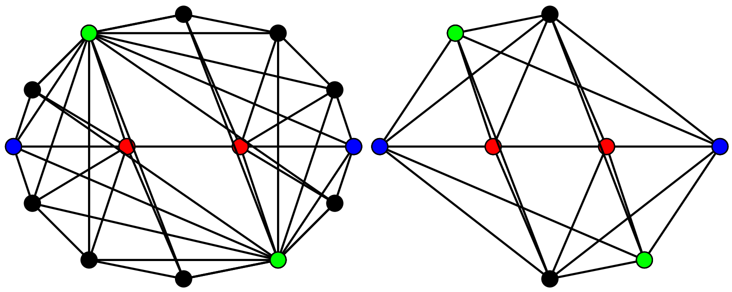


Let the two red vertices be c_1 and c_2 respectively, and $\delta(G) \geqslant 5$. Let all vertices in blue be $v_1, ...v_n$ and black vertices be $w_1, ..., w_n$. This implies that $\{v_1, ...v_n, w_1, ..., w_n\} \subset N(a) \cap N(b)$. If $N(a) \cap \{c_1, c_2\} = \emptyset$, then $G - b$ is planar. Similarly $N(b) \cap \{c_1, c_2\} = \emptyset$, due to symmetry. If, say, $c_1 \in N(a) \cap N(b)$, we get a K_6 minor, hence G is not minor minimal.

Proof: Continued



We show by contracting edges on the rim until we have a K_4 minor connected to a vertex of degree one (restricted to the DW_n graph). We contract this degree on vertex to either a or b and we have a K_6 minor. Let, with out loss of generality $N(a) \cap \{c_1, c_2\} = c_1$ and $N(b) \cap \{c_1, c_2\} = c_2$. Contract edges along the rim until we can form



Where vertices a and b, which are green, c_1 and c_2 in red, and the blue vertices will form a DW_4 subgraph. We contract edges for the remaining graph and yield a J graph. Hence G is not $MMNA$.

4.5 Conclusions

We will continue this further, into $|G| > 8$. Let it be the case that we continue until a patter of planar 3−connected graphs is discovered to finalize and prove the main conjecture, than we will have done what we set out to do. As such we conject the following.

Conjecture 6: 3−connected graphs generate $MMNA$

ll $MMNA$ graphs of connectivity $\kappa = 5$ can be generated from planar graphs of connectivity $\kappa = 3$.

Strategy

Moving Forward

We will continue to test increasingly larger sized graphs to determine underlying patterns of graphs to determine analytic methods for characterizing all $MMNA$ graphs of connectivity $\kappa = 5$.

Chapter 5

Conclusion and Questions

5.1 Introduction

Finding the complete list of Minor Minimal Not Apex graphs is a challenging problem. We have focused on the upper bound of connectivity for $MMNA$ graphs. The upper bound was shown to be for connectivity $\kappa = 5$. There are no graphs of connectivity $\kappa = 6$. As conjectured in chapter 3, we claim that there are only two $MMNA$ graphs of connectivity $\kappa = 5$. In the course of this research we have observed many supporting results.

5.1.1 Connecting K_6^- to $MMNA$ or $\kappa = 5$

The two conjectured $MMNA$ graphs of connectivity $\kappa = 5$ are K_6 and the Jørgensen graph, J. Interestingly both of these graphs have K_6^- as a minor. We also conjectured that if a graph is $MMNA$ with connectivity $\kappa = 5$ then it would have K_6^- as a minor. Further if K_6^- is a minor of a graph with the aforementioned properties, then the graph is either K_6 or J. We can extend this further by noting Bollobás we know that if a graph has $\delta \geqslant 5$ then it has either I_{12} or K_6^- as a minor. We saw that when a graph has $\delta \geqslant 5$ and the size of the graph is < 12, then it automatically has K_6^- as a minor. We extended this to graphs having 12 and 13 vertices with I_{12} as a minor and is also not-planar then we again have K_6^- as a minor.

Further Investigation

We believe that we can extend this idea to all graphs that have $\kappa \geqslant 5$ and are $MMNA$ will have K_6^- as a minor. We further this by showing in the following section the graph generator from K_4.

5.1.2 Building $\kappa = 5$ graphs from $3-$connected ones

As we have seen that Tutte shown all $3-$connected graphs can be constructed inductively through vertex splits. We saw that this was done by ensuring that all vertices splits have degree greater than or equal to three, and then maintaining that minimum degree. To do this we use the `plantri` package in SageMath to generate all planar graphs of vertex connectivity at least three and then adding two vertices. We ensured that $\delta \geqslant 5$ and check each graphs to ensure $\kappa \geqslant 5$. So far we saw that all graphs up to 8 vertices only have the two known $MMNA$ graphs of connectivity $\kappa = 5$, that is K_6 and J.

Further Investigation

We wish to iterate this further, with 9, 10,... vertices. We believe that we will see underlying patterns that will determine a proof for the main conjecture of this thesis.

5.2 Questions remaining

We still need to determine a proof for the main conjecture. To support this proof, we will likely find a proof for the supporting conjecture that is a graph is $MMNA$ and connectivitiy $\kappa = 5$ than it has K_6^- as a minor.

5.2.1 Proving K_6 and Jørgensen are the only $MMNA$ $\kappa = 5$ graphs

This proof is part of the overall proof looking for the complete list of $MMNA$ graphs, of which one direction for proof is to determine for each complete list per connectivity.

Appendix A

Definitions, Theorems, etc.

Appendix B

To be completed

To be completed . . .

Appendix C

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Graphs that are minor minimal not Apex of connectivity $\kappa \leqslant 5$

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