Finding a General Form for the A-Polynomial of Twist Knots

by

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Abstract

This work is concerned with the calculation of the A-polynomial for the family of hyperbolic twist knots K_n with n negative. It contains complete arguments for the p- and q-polynomials needed to calculate the A-polynomial for this family of hyperbolic knots along with conjectures about the general form for the A-polynomial modulo two when n is a negative power of two and the C-polynomial modulo two for all negative n.

Acknowledgement

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§1.0 Informal Introduction to Knot Theory

First and foremost, what is a knot, and—if you'll pardon the pun—what is not? In everyday life, we tend to think of knots as things people tie in rope or string in order to accomplish some task such as climbing a mountain or earning a merit badge. In fact, virtually everyone at some point in their lives learns to tie knots in some form or another. Furthermore, knots have also found extensive use in artwork. Books such as [Ba] document the beauty of the Celtic Knots. An example is found in Figure 1 below.

Figure 1. The Celtic "Infinity Dragon"

Although it might seem somewhat counter-intuitive, mathematically we wish to model knots as closed loops, that is, as loops without any beginning or end. An example is the light gray "infinity" loop in Figure 1 above. We do this instead of having an open loop (with the two ends left loose) as you might tie in a piece of string or rope. We also require that a knot does not intersect itself. An example of what many people would consider a knot appears in Figure 2 below. To turn this into a mathematical knot, though,

one would have to join the two ends at the top of the figure. By using such closed loops, we can employ pre-existing, powerful tools for studying knots. Specifically, with this model, we can distinguish knots using elements of topology and algebra. Some examples of mathematical knots appear in Figure 3 below.
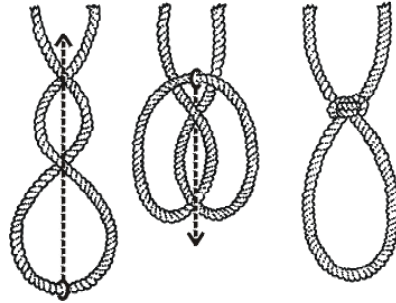
Figure 2. How to tie a Butterfly Noose, which is not a mathematical

knot because it doesn't form a closed loop.
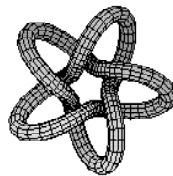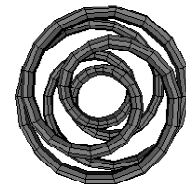
The figure-8   The star knot,   Another view of
knot, aka 4_1.   aka 5_1 in Tables.   the star knot.

Figure 3. Some mathematical knots, i.e., non-intersecting closed loops.

Knot Theory is the sub-field of algebraic topology that is primarily concerned with the problem of cataloging mathematical knots. As suggested in Figure 3 above, it can be difficult to recognize if two knots are the same just by looking at them. While

many authors ascribe the creation of Knot Theory to C. F. Gauss in the early 1800's, it was not until the efforts of scientists in the middle of the 1800's that the subject found much support in the scientific community.

Lord Kelvin in [Tho] postulated that one could explain the existence of the known distinct elements by modeling them as mathematical knots in the ether, which was at the time believed to permeate the universe. Specifically, he postulated that atoms were actually vortex loops, with different chemical elements consisting of different knotted configurations of ether. Thus, one way of categorizing the elements would be to start by describing all possible knots. For example, we might wish to associate the simplest of all elements with the simplest of all knots (called the *unknot*), as in Figure 4 below.

vs.
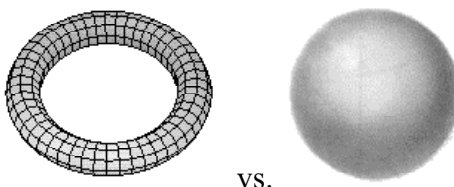
Figure 4. Is the unknot ether really the Hydrogen atom?

In order to pursue this hypothesis, scientists such as the Scottish physicist P. G. Tait and the American mathematician C. N. Little worked on the difficult problem of classifying knots entirely through trial and error. Tait's table can be found in [Ta]. Unfortunately, his catalogue of knots through ten crossings classified several knots incorrectly. However, with the Michelson-Morley experiment of 1887, this was inconsequential compared to the proven *non*-existence of ether! What's more, the advent of quantum mechanics followed shortly thereafter, and this type of applied knot theory

was quickly discarded by early twentieth century scientists in favor of quantum models for the chemical elements. Mathematicians such as C. N. Little, however, continued the study of knots as part of a newly emerging branch of mathematics known as *topology*.

Topology is often defined informally as "rubber sheet geometry" in the sense that objects are seen to be the same if they share certain intrinsic properties. [This is taken up more formally in Section 1.1 below.] Essentially, topologists see objects as if they were made out of highly malleable rubber that can be stretched and shrunk at will. Thus, from a topologist's eyes, two objects are seen to be the same if one can be continuously changed into the other without any funny business such as cutting or pasting. The classical—albeit still humorous—example of such topological equivalence found amongst everyday objects appears in Figure 5 below.
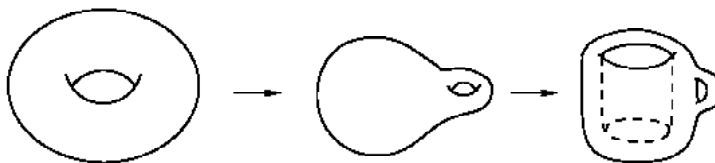
Figure 5. Topologically, a doughnut (a.k.a., a solid torus) is the same as a coffee cup.

Historically, topology began as a combination of the disciplines of analysis and geometry. Today, however, many mathematicians see the three major branches of mathematics as analysis, algebra, and geometry together with topology. *Algebraic topology* is essentially an attempt to apply ideas from algebra to topology. Indeed, the application of one branch of mathematics to another is often most fruitful. Unfortunately, though, even with the application of such powerful algebraic tools as homological

algebra and homotopy groups [described in Section 1.2 below], the problem of classifying knots remains unsolved.

At the same time, knot theorists have developed their own paraphernalia for this pursuit. One of the most useful is the notion of a *knot invariant*, informally something that does not change when we modify the knot's appearance without altering it topologically. There are countless examples of knot invariants, some of them borrowed from various other sub-fields of topology. Ideally, we would want any knot invariant to have two properties: it should be able to tell many different knots apart, and it should be easy to calculate.

Examples of Knot Invariants.

(1) *The Minimum Crossing Number of a Knot*: The minimum number of crossings needed to project a knot into the plane in order to create a two-dimensional diagram. While useful for organizing tables of knots such as the standard ones found in [R] or [Ad], there are many different knots with the same crossing number for more than four crossings. For example, there are 165 distinct prime knots [defined in Section 1.3] with 10 crossings. See Figure 6 below.

10_{160} $\qquad$ 10_{165}
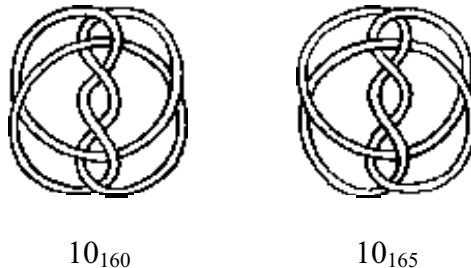
Figure 6. Two knots, each with ten crossings, from [R]. Notice that although they differ by only one crossing at the top of each diagram, they are still considered distinct knots.

(2) *The Bridge Number of a Knot*: The minimum bridge number of all possible diagrams of a given knot. The bridge number of a particular diagram is the number of maximal overpasses, where an overpass is a subarc of the diagram possessing an overcrossing but no undercrossing points. See [R] for more information. For the rest of this thesis, we will be primarily concerned with knots whose bridge number is two. These so-called *two-bridge knots* are described in detail in Section 1.3 below.

(3) *Polynomials*: There are many! This type of invariant has become very popular in recent years because polynomials are usually somewhat easy to calculate (with a notable exception being the A-polynomial), and they are typically very good at telling many different knots apart. In theory, they are really just fancy collections of numerical invariants such as the two examples given above. In practice, however, they encapsulate much more information about a knot than a single number can since, for example, we can apply the standard tools from analysis to them. Here are some of the most important developments in the history of knot polynomials:

- 1928: James Alexander introduces the first knot polynomial in [Al], now known as the *Alexander Polynomial*. It distinguishes many knots, but it is also somewhat difficult to calculate from Alexander's definition.

- 1969: John Conway simplifies the calculation of the Alexander Polynomial considerably using Skein relations.

- 1984: Vaughan Jones introduces a second knot polynomial in [J]. This is big news because the so-called *Jones Polynomial* is much better at telling

knots apart than the Alexander Polynomial, and it is fairly easy to calculate.

- 1984: Four months after Jones, many mathematicians simultaneously discovered what is now called the *HOMFLYPT Polynomial* and published it in [FYHLMO]. It turns out that this extremely robust polynomial generalizes both the Alexander and Jones Polynomials. Some mathematicians have described this publication as the start of "polynomial fever" among knot theorists.

- 1993: The *A-polynomial* is announced in [CCGLS]. While the above polynomials all describe various combinatorial aspects of knots, the A-polynomial encodes certain geometric information about the knot. For example, D. Boyd has hypothesized in [Bo] that the Mahler measure of the A-polynomial might be related to the hyperbolic volume of the knot. [See [Bo] for definitions of these terms.] However, while extremely good at telling knots apart, the A-polynomial is also extremely difficult to calculate. In fact, the 20-some knots for which the A-polynomial was tabulated in [CCGLS] remain the state of the art. A *Mathematica* program that we have used to calculate more than 30 A-polynomials for twist knots can be found in Appendix 4.1 below.

- 2001: The *C-polynomial* is announced in [Z]. Essentially, the C-polynomial is a simplified version of the A-polynomial, which we have also attempted to generalize for the infinite family of twist knots. A

Mathematica program that calculates such C-polynomials can be found in Appendix 4.5 below.

As you can see, mathematicians have done much with the theory of knots since others abandoned the subject in the late 1800's. However, the subject has now come full circle in a sense. There are many recently discovered applications of knot theory including the study of how DNA replicates. A good discussion of this appears in [Sum]. In addition, a more basic account of this and other newfound applications of knot theory in the biological and physical sciences can be found in [Ad] or [Sub]. These two references are also excellent resources for a more in depth introduction to knot theory than the one given above.

For the rest of this thesis, it is assumed that the reader is familiar with the content of undergraduate courses in real analysis, linear algebra, modern abstract algebra, and general topology. Good references for these subjects include [KF] for real analysis, [C] for linear algebra, [H] for modern abstract algebra, and [K] for general topology. As well, both [K] and [N] provide excellent introductions to algebraic topology.

§1.1 Introduction to Ambient Isotopies

In order to more formally study mathematical knots, we need to develop a small amount of algebraic topology for use in modeling them. Specifically, in this section we will more precisely define our chosen form of equivalence for knots by introducing first a homotopy and then a specific kind of homotopy known as an ambient isotopy.

The most basic way of defining topological equivalence is with a homeomorphism between two topological spaces of interest, say X and Y. The following definition defines an equivalence relation between topological spaces, as in [K].

Definition. A *homeomorphism* between X and Y is a continuous mapping $f: X \rightarrow Y$ that is a bijection of X onto Y with continuous inverse f^{-1}. The spaces X and Y are then said to be *homeomorphic*, and this notion of equivalence between topological spaces is denoted by $X \sim Y$.

However, this version of topological equivalence is not sufficient for the study of knots because under it every knot is homeomorphic to the *trivial knot* or *unknot*—i.e., the circle S^1 as it appears in the rightmost image of Figure 7 below. Figure 7 also contains an outline of how a trefoil knot can be continuously mapped onto the unknot.

Figure 7. A Homeomorphism: Cut a knot, rearrange it, and then glue it back together.

To get an appropriate definition of equivalence, we will need more than simply the idea of a continuous mapping between two spaces; we will need to consider the way the knots are embedded in their ambient space. A first step in this direction is to introduce a somewhat simple idea via a convoluted definition. This is typical in mathematics: often the harder the definition, the more useful it is. A good example of this phenomenon is the so-called ε-δ definition of a limit in analysis. However, as L. C. Kinsey promises in Section 9.1 of [K], "You will end up liking homotopies. Trust me."

Throughout, X and Y are Hausdorff spaces, and $I = [0, 1]$ denotes the *unit interval*.

Definition. Let $f: X \rightarrow Y$ and $g: X \rightarrow Y$ be continuous mappings. Then a *homotopy* from f to g is a continuous mapping $h: X \times I \rightarrow Y$ for which $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$, for each $x \in X$. The mappings f and g are then said to be *homotopic*, denoted $f \sim g$.

Examples. There are many standard examples of homotopies.

(1) Visually, a homotopy can be seen as a continuous transformation between the images of two functions, where the second argument of h, namely t, measures the amount of time that has elapsed from the start to the finish of this transformation. An example appears in Figure 8 below where only a few of the uncountable number of intermediate steps from $f(X)$ to $g(X)$ are shown.
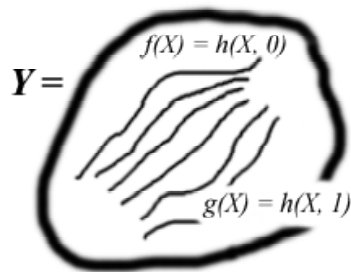

Figure 8. A few steps in the homotopy from f to g in "Trilobite space" Y.

(2) A more intuitive example of a homotopy can be found in the gradual transformation of the spending function of a college student from when one starts college to when one gets a real job. A typical poor starving student is usually very frugal, but gradually spending increases until a real job is obtained upon graduation. Then spending habits tend to become more frivolous due to this newfound wealth from gainful employment.

As with the notion of topological spaces being homeomorphic, "homotopic" is also an equivalence relation on the set of all continuous maps between two topological spaces. A proof of this can be found in Section 9.2 of [K]. Furthermore, we will now, in a manner of speaking, combine the two ideas in order to properly define how we wish to view knots as equivalent.

For a set E, let $\text{id}_E: E \to E$ denote the *identity function* on E. That is,

$$\text{id}_E(e) = e, \text{ for each } e \in E$$

Definition. Let $f: X \to Y$ and $g: X \to Y$ be continuous mappings. Then an *ambient isotopy* from f to g is a homotopy $h: Y \times I \to Y$ for which

(1) $h(\bullet, t)$ is a homeomorphism (from Y to itself) for each $t \in [0, 1]$

(2) $h(\bullet, 0) = \text{id}_Y$

(3) $h(f(x), 1) = g(x)$ for each $x \in X$.

The mappings f and g are then said to be *ambient isotopic*, and this is denoted $f \approx g$.

In other words, h is a homotopy (through homeomorphisms of Y) between the mapping id_Y and a homeomorphism of Y that takes the image of f to the image of g.

Examples. As with homotopies, there are many standard examples of ambient isotopies.

(1) Since an ambient isotopy h from f to g is really just a special homotopy, we can again think of h as transforming the image of f into the image g. In this sense, h actually warps the ambient space containing the image $f(X)$ so that $f(X)$ becomes $g(X)$. For example, suppose we have functions f and g that describe how to embed a doughnut and a coffee cup, respectively, into 3-space, $\mathbb{R}^3$. Then the changing of one into the other by h as in Figure 5 above is an example of an ambient isotopy. In this case, $h: \mathbb{R}^3 \times I \to \mathbb{R}^3$.

(2) The manipulation of a knot made out of string that is done without any cutting or gluing is an ambient isotopy. We will see in Section 1.3 that a knot is an embedding of a circle, i.e., S^1, into the 3-sphere, S^3. However, it is again convenient to just think of the knot as the image of this embedding. See Figure 9 below, where X is the 1-sphere S^1 knotted into the figure-8 knot and Y is the ambient space, S^3. (We explain in Section 1.3 below why we choose this as the ambient space.) This demonstrates that for knots, an ambient isotopy essentially distorts the ambient space, taking the knot along with it. The net effect is the transformation of one representation of a knot into another in a way that naturally models how we manipulate a real knot made out of string.
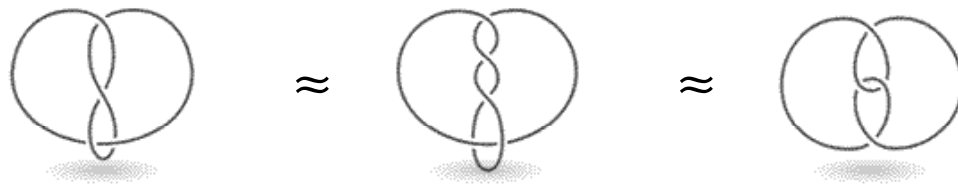
Figure 9. Three different ambient isotopic views of the figure-8 knot.

§1.2 Definition of the Fundamental Group

Now that we have the concept of functions being homotopic, in this section we will motivate the definition of the A-polynomial by introducing the concept of the fundamental group of a pathwise connected topological space. Specifically, the A-polynomial is based on the fundamental group of the knot's complement, that is, the manifold formed by removing the knot from its ambient space.

Throughout X is a pathwise connected Hausdorff space, and I denotes the unit interval. We begin with two important preliminary definitions:

Definition. A *loop* in space X that is based at the point $x \in X$ is a continuous mapping $\gamma : I \to X$ for which the so-called *end points* are both x, that is, $\gamma(0) = \gamma(1) = x$.

Definition. Two loops α and β in X that are both based at the same point $x \in X$ are said to be *path homotopic* if there is a homotopy $h : I \times I \to X$ from α to β for which the end points remain fixed at x, that is,

$$h(0, t) = h(1, t) = x \text{ for all } t \in I.$$

Thus, as t varies from zero to one, h is allowed to stretch and twist at will as along as it remains continuous. No cutting or pasting of the intermediate loops is allowed in this transformation of α into β so that the transition between intermediate loops is continuous.

In [K] the author proves that path homotopy is an equivalence relation between loops in X. Thus, for a given loop α in X, we will denote the equivalence class generated by α as

$$[\alpha] = \{\beta \mid \beta \text{ is a loop in X path homotopic to } \alpha\}$$

In fact, we can even build an operation between two paths that is preserved under this notion of path homotopy so that a group of these equivalence classes can be formed. The resulting group is called the *first homotopy group* or the *fundamental group* of a connected topological space X, and it is an extremely powerful tool in topology. A proof that the structure defined below is a group can be found in [K].

We begin with a natural definition for combining loops:

Definition. For two loops α and β in X that are both based at a point $x \in X$, we define their *product* to be the loop

$$\alpha \bullet \beta = \begin{cases} \alpha(2t), & 0 \le t \le 1/2 \\ \beta(2t-1), & 1/2 \le t \le 1 \end{cases}$$

Essentially, this definition combines the two loops α and β by telling us to traverse loop α in double-time, followed immediately by loop β in a similar fashion. The end result is to traverse both loops in exactly the same amount of time it takes to traverse either one of the loops separately. It is proven in [K] that this operation is both well-defined and associative up to a path homotopy.

Next, we define the notion of inverting a loop, also in a natural manner.

Definition. For a loop $\alpha : I \to X$ based at point $x \in X$, we define the *inverse* of α to be the loop $\alpha^{-1} : I \to X$, also based at the point $x \in X$, and defined by $\alpha^{-1}(t) = \alpha(1-t)$.

One can think of α^{-1} as traversing the loop α in the opposite direction. Furthermore, under the product given above, we can define an identity loop $\iota : I \to X$ based at a point $x \in X$ by not going anywhere. That is, we define $\iota(t) = x$ for each $t \in I$. It is proven in [K] that not only is the inverse of a loop defined above unique up to path

homotopy, but also that the product of a loop and its inverse is path homotopic to the

identity loop.

We now have all the ingredients for our group:

Definition. Let $x \in X$. Then the *fundamental group* of X (with respect to the point

x) is the set

$$\pi_1(X, x) = \{[\alpha] \mid \alpha : I \to X \text{ is a loop in } X \text{ based at } x\}$$

together with the binary operation $[\alpha] \cdot [\beta] = [\alpha \bullet \beta]$, for all $[\alpha]$, $[\beta] \in \pi_1(X, x)$.

Furthermore, the equivalence class $[\iota]$ acts as the identity element of $\pi_1(X, x)$, and for

each equivalence class $[\alpha] \in \pi_1(X, x)$, the inverse of $[\alpha]$ is the class $[\alpha]^{-1} = [\alpha^{-1}]$.

Examples. The fundamental group, as its name suggests, has many important

applications. Below are some elementary examples and results, some of them related to

knot theory.

(1) It is a basic result of algebraic topology proven in [K] that the calculation of

the fundamental group is independent of the chosen base point, up to an

isomorphism. In other words, let X be a pathwise connected Hausdorff space,

and let x_1, $x_2 \in X$. Then the fundamental groups are isomorphic. We'll denote

this by

$$\pi_1(X, x_1) \approx \pi_1(X, x_2)$$

Thus, we will generally not specify a base point when describing a

fundamental group.

(2) The fundamental group of the circle S^1 is isomorphic to $\mathbb{Z}$ under addition. To

see this, notice that if we fix a point $x \in S^1$, then there are only two possible

non-homotopic loops that traverse the circle S^1 once. One of these loops, μ,

travels along S^1 clockwise, and the other one, μ^{-1}, travels in the opposite direction, counterclockwise. Furthermore, we may traverse the circle any number of times in either direction, while traversals of the form $\mu^n \mu^{-n}$ (n an integer) are all trivial. Thus, we may use either of the two initial loops to generate a free abelian group $\pi_1(S^1) = <\mu>$ on the single generator μ, which is then isomorphic to $\mathbb{Z} = <1>$ as desired. A more formal proof can be found in any advanced text on algebraic topology such as [Mas].

(3) Another basic result of algebraic topology is that if X and Y are pathwise connected Hausdorff spaces, then the fundamental group of their product space is isomorphic to the direct product of their fundamental groups, i.e., $\pi_1(X \times Y) \approx \pi_1(X) \oplus \pi_1(Y)$. Recall that the torus, T^2, is the surface formed by taking the product space of S^1 with itself. Thus, from example (2) above,

$$\pi_1(T^2) = \pi_1(S^1 \times S^1) \approx \pi_1(S^1) \oplus \pi_1(S^1) \approx \mathbb{Z} \oplus \mathbb{Z}.$$

In other words, the fundamental group of a torus is a free abelian group on two generators. Using a standard embedding of T^2 into $\mathbb{R}^3$, we can present this group as $\pi_1(T^2) = <\mu, \lambda>$, where μ is a *meridian curve* of the torus and λ a *longitude curve*. See Figure 10 below.
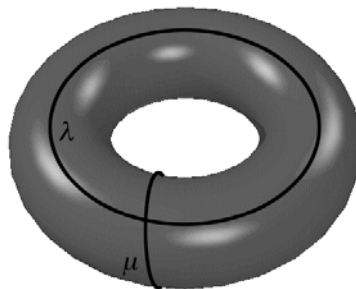
Figure 10. A Torus with generating loops μ and λ drawn.

(4) Recall that the knot complement is the manifold formed by removing the knot from its ambient space, which we take to be S^3. In 1908, H. Tietze conjectured that the complement of the knot is a full knot invariant, i.e., that no two distinct knots can have homeomorphic complements. This was proven for what are called prime knots [defined in Section 1.3 below] by C. Gordon and J. Luecke in [GL]. The fundamental group of the knot's complement is also a knot invariant. In other words, ambient isotopic knots will have homeomorphic complements and those complements in turn will have isomorphic fundamental groups. The isomorphism between these fundamental groups is said to be *induced* by the homeomorphism between the complements, and its existence is proven in [Mas].

§1.3 Formal Knot Theory Overview

We begin by formally defining a knot:

Definition. A *knot K* is a continuous embedding of the 1-sphere S^1 into the 3-sphere S^3. That is, a knot is a one-to-one continuous mapping $K: S^1 \hookrightarrow S^3$. Furthermore, with each knot K we associate the fundamental group $\pi_1(V)$ of the knot's complement, $V = S^3 \setminus K$, and we call it the *group of the knot* or the *knot group*, typically denoted by abuse of notation as $\pi_1(K)$.

Recall that $S^3 = \mathbb{R}^3 \cup \{\infty\}$. Since we can always rearrange a knot with an ambient isotopy to avoid the so-called point at infinity, it is often convenient to think of the knot as being in $\mathbb{R}^3$, especially since it's very difficult to imagine four-dimensional space. However, it is also very convenient to have the ambient space be compact as is S^3, and many results in knot theory are specific to the S^3 case. See [R] for examples.

There are many ways of cataloguing knots, most of which rely on some knot invariant such as the fundamental group. An important classification theorem due to W. Thurston sorts knots into one of three distinct categories: torus knots, satellite knots, and hyperbolic knots. For a proof, see [Thu 82] and [Thu 86].

Definition. A (p, q)-*torus knot K*, for p and q non-negative, relatively prime integers, is one that passes p times around the meridian and q times around the longitude of the standard embedding of a torus.

Recall that the fundamental group of the standard embedding of a torus T can be presented as $\pi_1(T) = <\mu, \lambda>$ (see example (3) of the previous section). A (p, q)-torus knot

represents the equivalence class $[\mu^p \lambda^q]$ in $\pi_1(T)$.

Examples. There are obviously an infinite number of torus knots. They are determined by the pair $\{p, q\}$, except for the unknot which can be represented as $(1, n)$ or $(n, 1)$ for any non-negative integer n. Below are two standard examples and one more fancy example. For more examples, see [Ad].

(1) The *unknot*, the trivial loop with no knotting in it, is the $(1, 0)$-torus knot (though, as above, there are many other representations). See Figure 11 below.
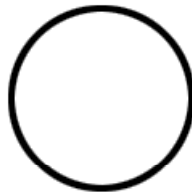
Figure 11. The unknot is the $(1, 0)$-torus knot.

(2) The *trefoil knot* 3_1 is the $(3, 2)$-torus knot. See Figure 12 below.

Figure 12. The trefoil knot is the $(3, 2)$-torus knot.

(3) The illustrious *Solomon's seal knot* 5_1, also called the *star knot*, is the $(5, 2)$-torus knot. See Figure 13 below.

Figure 13. Solomon's seal knot is the (5, 2)-torus knot.

One useful way of combining knots is to essentially knot a knot. This gives the important category of satellite knots.

Definition. Let K_1 be a knot inside a torus T, and embed T into S^3 as a second knot K_2. Then the resulting knot K_3 is the *satellite knot* of K_1 with the *companion knot K_2*.

Examples. An important method for generating satellite knots is by taking the *connected sum* of two knots, K_1 and K_2, denoted $K_1 \# K_2$. (This is also often called the *knot sum*.) Essentially, this amounts to cutting the two knots and gluing the resulting two loose ends of one knot to those of the other knot. While is it not immediately obvious that taking the connected sum of two knots yields a satellite knot, a proof can be found in [R]. See Figure 14 below for an example. For further examples, see [Ad].
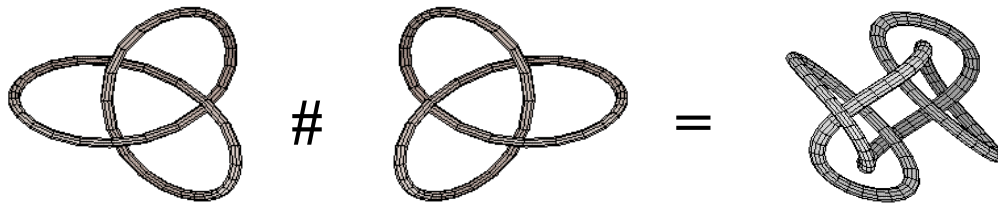
Figure 14. The connected sum of two trefoil knots gives

a satellite knot, called the *Square knot*.

The knot resulting from the connected sum of two nontrivial knots is called a

composite knot. In a similar manner we then define a *prime knot K* to be a knot with the property that if $K = K_1 \# K_2$, then either K_1 or K_2 is the unknot. There are other satellite knots besides those that can be formed from taking the connected sum of two knots, and it turns out that there an infinite number of both prime and composite satellite knots.

Finally, we have one last type of knot under Thurston's trichotomy.

Definition. A *hyperbolic knot* is one that is neither a torus knot nor a satellite knot.

The name "hyperbolic knot" reflects the more common definition, being a knot whose complement admits a hyperbolic metric. See [Ad] for a detailed explanation of what this means. However, in light of Thurston's result, it is somewhat more convenient to define a hyperbolic knot to be one that is neither a torus nor a satellite knot as above.

At the same time, since there are infinitely many torus knots and infinitely many satellite knots, one is left to wonder how many knots could possibly fall into the third category of hyperbolic knots. However, all but two of the twist knots [defined in Section 2.1 below] are hyperbolic. In fact, Thurston has provided proof in [Thu 82] and [Th 86] that "almost all" prime knots are hyperbolic. Specifically, he has demonstrated that the infinitudes of the torus and prime satellite knots are actually asymptotically insignificant (for example, as the crossing number tends to infinity) compared to the number of hyperbolic knots. A more complete discussion of this can be found in [Ad].

Example. A somewhat *ad hoc* demonstration of the fact that almost all knots are hyperbolic is found in the prime knot tables of [R] and [Ad]: All but five of the sixty-seven knots with nine or fewer crossings found in [Ad] are hyperbolic! These non-hyperbolic knots with small crossing number are denoted by convention as 3_1 (the

trefoil), 5_1 (Solomon's seal knot), 7_1, 8_{19}, and 9_1. For examples of hyperbolic knots with a more complicated structure than the one given in Figure 15 below, see [Ad].

Figure 15. The figure-8 knot is the simplest hyperbolic knot.

Another important invariant used in the process of classifying knots is the bridge number of a knot. For example, there is only one knot with bridge number 1, namely the unknot as shown in Figure 11 above. The next natural bridge number to investigate is much less trivial. In fact, the so-called *two-bridge knots*, i.e., those knots with bridge number two, form a very important collection of knots. They can essentially be characterized as having a projection for which a plane divides the knot into two separate collections of non-intersecting arcs. A projection for the figure-8 knot with this property is given in Figure 16 below, with the bisecting plane lying below the two vertical arcs (called the *bridges*) and above the curved arcs.
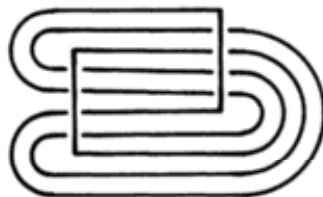
Figure 16. A projection of the figure-8 knot that can be split into non-intersecting arcs with a single plane, demonstrating that the figure-8 knot is a two-bridge knot.

When we describe the fundamental group of knot, as we will for the two-bridge knots below, it is often presented in terms of meridians and longitudes of the knot.

Definition. Let K be a knot with complement V. Then a *meridian* μ of K is the class in $\pi_1(V)$ generated by a simple closed curve in V that bounds a punctured disc in V orthogonal to the original knot. A *longitude* λ of K is the class in $\pi_1(V)$ generated by a simple closed curve in V, which together with a meridian μ, forms a basis for $\pi_1(T)$, the fundamental group of the torus, T, that "follows" the knot.

The idea of T "following" the knot can be made rigorous. For example, T is the boundary of a tubular neighborhood of the knot (see [R]). Informally, if we thicken the 1-dimensional curve representing the (image of the) knot, we obtain a solid torus, $S^1 \times D^2$. Then T is the boundary of this solid torus. We will continue to use the terminology of T "following" K to describe this situation.

Example. Figure 10 above shows a standard embedding of a torus that follows the unknot. A meridian of the torus is the edge of a disc orthogonal to the knot and therefore also a meridian of the knot. Similarly, a longitude of the torus is also a longitude of the knot. Informally, meridians encircle the knot once while longitudes are homotopic to the knot.

One property that makes the two-bridge knots a useful set of examples is that the structure of their fundamental groups is very well understood. As proven in [Bu], if K is a two-bridge knot, then $\pi_1(K) = \langle x, y \mid x\, w\, y^{-1}\, w^{-1}\rangle$ where x and y are meridians of K and

$$w = x^{\varepsilon_1} y^{\varepsilon_2} \ldots y^{\varepsilon_{\alpha-2}} x^{\varepsilon_{\alpha-1}}$$

is called the *word* of the group. Here α is an odd positive integer greater than one, β is an odd integer that is relatively prime to α and satisfies the inequality $-\alpha < \beta < \alpha$, and

$$\varepsilon_k = (-1)^{\left\lfloor \frac{k\beta}{\alpha} \right\rfloor} \text{ for each } 1 \le k \le \alpha - 1$$

where $\lfloor \bullet \rfloor$ denotes the greatest integer function. In fact, this presentation of $\pi_1(K)$ is *symmetric* in the sense that $\varepsilon_k = \varepsilon_{\alpha-k}$ for each $1 \le k \le \alpha - 1$. Often we will refer to a two-bridge knot by the pair (α, β) following [Sc], in which two-bridge knots were first introduced.

An important class of two-bridge knots consists of the twist knots, defined in Section 2.1 below.

§2.0 Introduction to the Problem

In [CCGLS], the authors introduce a new knot invariant called the A-polynomial and then remark that the "computations are especially tractable in the case of 2-bridge knots." The problem we have spent approximately one year working on involves attempting to generalize the A-polynomial for the very special sub-family of the two-bridge knots known as the twist knots. Even then, we have only been able to produce results in finding the general form of the A-polynomial for twist knots with negative tangle number.

In the meantime, we have also looked at various other characteristics of these twist knots. For example, we spent much of Summer 2001 studying the related C-polynomial, introduced in [Z]. During the Fall of 2001, we examined the connection between the Mahler measure of A-polynomials of twist knots and their hyperbolic volume. Around this time, we observed a general form for the A-polynomial of -2^n twist knots. Finally, during the Spring 2002, we have tried various techniques to prove the validity of our observation. These techniques include matrix algebra, subresultant chains, and polynomial remainder sequences. *Mathematica* programs that generate pseudo-polynomial remainder sequences and subresultant chains can be found in Appendix 4.4 below. Consult [Lo] for the algorithms implemented in these programs.

§2.1 Definition of Twist Knots

In this section we give a definition of the twist knots and then provide a few recognizable examples.

Definition. For an integer n, the *twist knot K_n* is the knot that can be drawn with the form given in Figure 17 below, with the convention that the crossings are oriented as in Figure 18 below. We call the number n the *tangle number* of K_n.
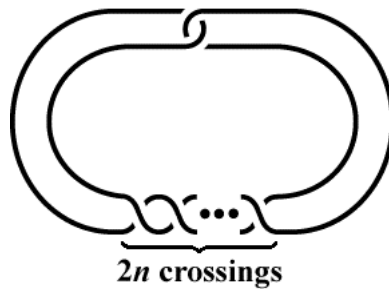
2n crossings

Figure 17. The Twist Knot K_n.

Positive Crossing   Negative Crossing

Figure 18. A positively oriented crossing versus a negatively oriented crossing.

Examples. Below are some twist knots that you should recognize.

(1) If the tangle number is zero, then the knot K_0 has only two total crossings. However, a knot with two crossings is obviously just the unknot. See Figure 19 below.
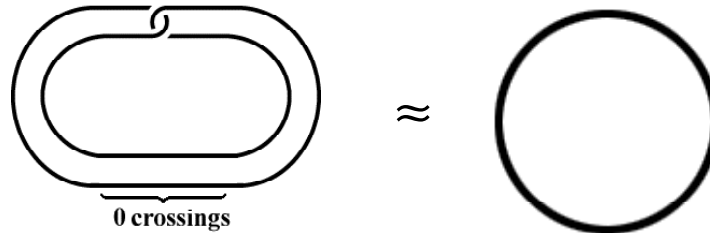
Figure 19. The Twist Knot K_0 is the unknot.

(2) If the tangle number is positive one, then the knot K_1 has four crossings total. Upon rearrangement, we see that K_1 is really the trefoil knot. See Figure 20 below.
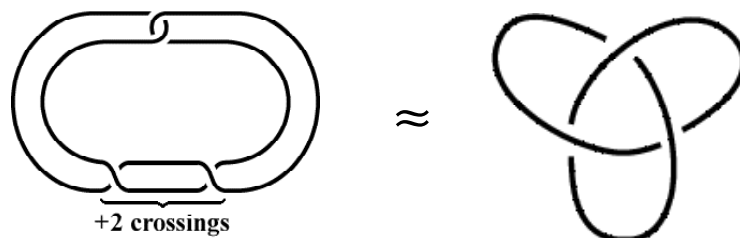
Figure 20. The Twist Knot K_1 is the trefoil knot.

(3) If the tangle number is negative one, then the knot K_{-1} has four crossings total. Upon rearrangement, we see that K_{-1} is really the figure-8 knot. See Figure 21 below.
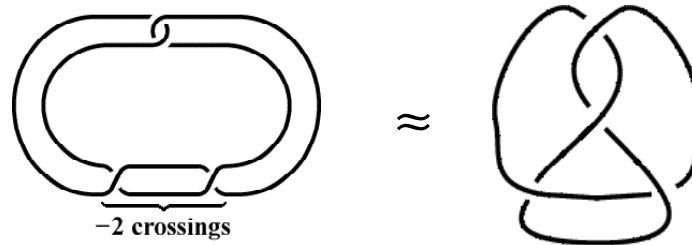
Figure 21. The Twist Knot K_{-1} is the figure-8 knot.

It is proven in [R] that the twist knot K_n is either the two-bridge knot $(4n-1, 4n-3)$, if n is positive, or the two-bridge knot $(-4n+1, -4n-1)$, if n is negative. This fact is use extensively in Section 3 below.

§2.2 Definition of the A-polynomial

The A-polynomial is a two-variable polynomial knot invariant first introduced in 1994 in [CCGLS]. Since this time, many A-polynomials for various knots have been tabulated, but no one has yet calculated this invariant for an infinite family of knots. Quite simply, it's hard!

First, some preliminary definitions. Throughout, when we say that a rational polynomial is irreducible, we mean that it cannot be factored over the field of rational numbers.

Definition. An *algebraic set S* in $\mathbb{C}^n$ is the set of all zeros of a finite collection of rational polynomials P in n indeterminates. In other words, $(r_1, \ldots, r_n) \in S$ if for all polynomials $p \in P$ we have that $p(r_1, \ldots, r_n) = 0$. Furthermore, S is called *irreducible* if it is the set of all zeros of a single irreducible rational polynomial p in n indeterminates.

It can be shown that any algebraic set S can be written as the union of a finite number of irreducible algebraic sets, called the *irreducible components* of S. See [La] for a more complete discussion. Furthermore, for any algebraic set, we can find a single polynomial that in a sense "defines" the set.

Definition. A *defining polynomial* for an algebraic set S is any rational polynomial that vanishes exactly on S and has no repeated irreducible factors.

One of the first difficulties in studying the A-polynomial is sorting through the varying definitions of it found in the literature, such as in [BZ], [CCGLS], [CL 96], and [D]. We wish to define the A-polynomial as the defining polynomial of an algebraic set

in $\mathbb{C}^2$. The definition given below is based on [CL 96], which simply multiplies the

defining polynomials associated with each irreducible component. This differs from the

original definition given in [CCGLS], in which the authors count multiplicities of the

irreducible components so that the A-polynomial can have repeated factors. It is proven

in [CCGLS] that the polynomial associated with each component is unique up to a

constant multiple.

Definition. Let K be a knot with complement V whose fundamental group is

given by $\pi_1(V)$. Fix a meridian-longitude basis of $\pi_1(T) = \langle\mu, \lambda\rangle$, where T is a torus that

follows the knot K. Let $R = \text{Hom}(\pi_1(V), \text{SL}_2(\mathbb{C}))$ be the set of all homomorphisms from

$\pi_1(V)$ to $\text{SL}_2(\mathbb{C})$, the group of 2×2 complex matrices having determinant one under matrix

multiplication. Let $R_u \subset R$ consist of those $\rho \in R$ for which both $\rho(\mu)$ and $\rho(\lambda)$ are upper

triangular. Then we define an *eigenvalue map* $\xi \equiv (\xi_\lambda \times \xi_\mu) : R_u \to \mathbb{C}^2$ by $\xi(\rho) = (L, M)$

where L is the upper-left entry of $\rho(\lambda)$ and M is the upper-left entry of $\rho(\mu)$. The closure

of the image of R_u under ξ, $\overline{\xi(R_u)}$, consists of several irreducible algebraic components in

$\mathbb{C}^2$. The product of the defining polynomials for these distinct algebraic components is

defined to be the *A-polynomial* of K, denoted A_K.

Remark. In other words, a pair of complex numbers, (L, M), will satisfy the

equation $A_K(L, M) = 0$ if there is a $\rho \in R$ for which

$$\rho(\lambda) = \begin{pmatrix} L & * \\ 0 & 1/L \end{pmatrix} \text{ and } \rho(\mu) = \begin{pmatrix} M & * \\ 0 & 1/M \end{pmatrix}$$

However, by abuse of notation, L and M will represent both these eigenvalues of

$\rho(\lambda)$ and $\rho(\mu)$, respectively, and the indeterminates of the A-polynomial. Moreover, we

will often refer to L and M as the longitude and meridian, respectively. The context will make the usage clear.

Since the polynomial associated with each component is unique up to a constant multiple, a suitable constant can be found so that $A_K(L, M) \in \mathbb{Z}[L, M]$. Moreover, we will also require that the A-polynomial contain no overall integral constant. As such, the A-polynomial is defined up to sign. Another important observation about the A-polynomial proven in [CCGLS] is that all of the powers of M in $A_K(L, M)$ will be even.

Examples. The A-polynomials for the first few twist knots are given below.

(1) The unknot K_0 has A-polynomial

$$A(L, M) = L - 1$$

(2) The trefoil knot K_1 has A-polynomial

$$A(L, M) = (L-1)(1 + LM^6)$$

(3) The figure-8 knot K_{-1} has A-polynomial

$$A(L, M) = (L-1)(L - LM^2 - M^4 - 2LM^4 - L^2 M^4 - LM^6 + LM^8)$$

§2.3 Formulation of the A-polynomial for Twist Knots

As mentioned in Section 2.0 above, in this section we will develop the method outlined in Section 7 of [CCGLS] to calculate the A-polynomial, $A_{-n}(L, M)$, for the twist knot K_{-n}, with n a positive integer. Recall that L and M represent both the eigenvalues of the longitude and the meridian of the knot under the homomorphism ρ and the indeterminates of the A-polynomial.

Before we begin, we will need to look more closely at the set R given in the definition of the A-polynomial. Elements ρ of R are often called *representations* of the knot group. We distinguish between two important types of representations.

Definition. A *reducible representation* of a knot group is one whose image in $SL_2(\mathbb{C})$ can be conjugated so that it lies in the set of upper triangular matrices. An *irreducible representation* is then one that is not reducible.

Representations of a group can be defined in a much broader sense. However, it should be noted that our definition agrees with the more general definition of a reducible representation in terms of invariant subspaces. See [La] for more information.

Essentially, we will build A_{-n} by choosing a particularly nice family of irreducible representations for the knot group in $SL_2(\mathbb{C})$. Then from the image of this family of irreducible representations under the eigenvalue map into $\mathbb{C}^2$ (parameterized by the coordinates L and M), we construct the A-polynomial $A_{-n}(L, M)$ as the defining equation for the irreducible algebraic component generated by this image. Due to the chosen representations, this amounts to finding two polynomials, p and q, so that simultaneously

solving $p = 0$ and $q = M^2L$, will both satisfy the group relator and yield representations having longitude eigenvalue L.

We begin with the usual presentation of the fundamental group of K_{-n}, as given in the Section 1.3 above:

$$\pi_1(K_{-n}) = <x, y \mid x \, w \, y^{-1} \, w^{-1}>$$

where x and y are meridians of K_{-n} and w is the word

$$w = x^{\varepsilon_1} \, y^{\varepsilon_2} \cdots y^{\varepsilon_{\alpha-2}} \, x^{\varepsilon_{\alpha-1}}$$

As discussed in Section 2.1, for the twist knot K_{-n}, $\alpha = 4n + 1$ and $\beta = 4n - 1$ so that

$$\varepsilon_k = (-1)^{\left\lfloor \frac{k\beta}{\alpha} \right\rfloor} = (-1)^{\left\lfloor \frac{4n-1}{4n+1}k \right\rfloor} \text{ for each } 1 \le k \le 4n - 1$$

where $\lfloor \bullet \rfloor$ denotes the greatest integer function.

Now, following [CCGLS], we define a particularly useful family of representations for the knot group $\pi_1(K_{-n})$. The reducible representations of $\pi_1(K_{-n})$ are well understood. As proven in [CCGLS], these reducible representations have the image $L = 1$ in $\mathbb{C}^2$ under the eigenvalue map. In other words, $\{L : L - 1 = 0\}$ is a component of $\overline{\xi(R_u)}$ for every knot, and if ρ is reducible, then $\xi(\rho)$ will lie within this component. For example, the A-polynomial of the unknot is $L - 1$ as the knot group, $\pi_1(K_0) \approx \mathbb{Z}$, has only reducible representations into $SL_2(\mathbb{C})$. This follows from Jordan's Decomposition Theorem for complex matrices, since the image of the generator for $\mathbb{Z}$ can be made upper triangular and since the product of two upper triangular matrices is upper triangular.

For the special case of the twist knots, it is proven in [Bu] that $\overline{\xi(R_u)}$ will only contain two components. One, which comes from reducible representations, will have $L - 1$ as its defining polynomial. Thus, the A-polynomial will be $(L - 1)$ times a single

irreducible polynomial. However, we will generally not mention this extraneous factor so

that when we give explicit examples of A-polynomials for the twist knots, we will

mention only the irreducible factor arising from irreducible representations.

As such, we are interested in studying components of $\overline{\xi(R_u)}$, other than $\{L : L - 1$

$= 0\}$, which, therefore, must contain points coming from irreducible representations.

Thus, we will investigate irreducible representations ρ, of the knot group $\pi_1(K_{-n})$. Since

$$x = w\, y\, w^{-1}$$

x and y are conjugate in $\pi_1(V)$, and so $\rho(x)$ and $\rho(y)$ are conjugate in $\mathrm{SL}_2(\mathbb{C})$ as well.

Furthermore, since conjugation does not change the eigenvalues of a complex matrix,

$\rho(x)$ and $\rho(y)$ must have the same eigenvalues, and we may assume that $\rho(x)$ is upper

triangular by Jordan's Decomposition Theorem for complex matrices. Thus, since ρ is an

irreducible representation, $\rho(y)$ cannot also be upper triangular since if it where, then ρ

would be reducible. However, it is proven in [CCGLS], that $\rho(y)$ can be made lower

triangular. As such, we map the generators of $\pi_1(K_{-n})$ to the matrices

$$\rho(x) = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix} \text{ and } \rho(y) = \begin{pmatrix} M & 0 \\ t & M^{-1} \end{pmatrix}$$

where M and t are indeterminates which parameterize our family of representations. Note,

however, that not every substitution of complex values for M and t will cause ρ to be a

homomorphism. Thus, we need to determine the subset of $\mathbb{C}^2$ (parameterized by M and t),

for which ρ is a homomorphism.

For the twist knot K_{-n}, we prove in Section 3.4 below that the matrix resulting

from the group relator,

$$\rho(x\, w\, y^{-1}\, w^{-1}) = \rho(x\, w) - \rho(w\, y)$$

has the form

$$\begin{pmatrix} 0 & (M^{-2})p(M,t) \\ (-M^{-2}t)p(M,t) & 0 \end{pmatrix}$$

where p is an integral polynomial in M and t. Thus, setting $p(M, t) = 0$ will make sure that ρ satisfies the group relator (and is thus a homomorphism), since $\rho(x\, w\, y^{-1}\, w^{-1})$ will then be the zero matrix.

Similarly, we must find a nice image for the longitude L. As proven in [BuZ], the longitude λ of any two-bridge knot is in the second commutator subgroup of $\pi_1(V)$, and so it should have zero exponent sum. Furthermore, as proven in [CCGLS], $\lambda = x^z ww^*$, where w^* is the word w written in reverse order and z is chosen so that the exponent sum is zero. However, as proven in Section 3.1 below, $z = 0$ for our K_{-n}. Furthermore, as shown in Section 3.5 below, the upper left entry of $\rho(ww^*)$ has the form $q'(M, t)/M^2$ where $q'(M, t)$ is an integral polynomial in M and t. Thus, since ww^* is the word representing the longitude, setting $q(L, M, t) = q'(M, t) - M^2 L = 0$ will give a representation whose longitude eigenvalue is L.

Before proceeding with the determination of the A-polynomial from these polynomials p and q, we will need the following definitions and results:

Definition. Let $f(x) = a_0 + a_1 x + \dots + a_n x^n$ and $g(x) = b_0 + b_1 x + \dots + b_m x^m$ be complex polynomials in the indeterminate x, with degrees n and m, respectively. Then the *resultant* of f and g, with respect to x, is the product

$$R(f, g) = \prod_{i=1}^{n} \prod_{j=1}^{m} (\beta_j - \alpha_i)$$

where $\alpha_1, \dots, \alpha_n$ are the n roots of f and $\beta_1, \dots, \beta_m$ are the m roots of g.

The resultant of two polynomials, f and g, is a way of determining if f and g have any roots in common. This is evident from the defintion above since $R(f, g) = 0$ if and only if $\beta_j = \alpha_i$ for some $1 \le i \le n$ and $1 \le j \le m$. However, if the roots of f and g are not known, a basic result of classical elimination theory shows that the resultant of f and g can be found by taking the determinant of the so-called *Sylvester matrix*, i.e.,

$$R(f,g) = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_o & & & & \\ & a_n & a_{n-1} & \cdots & a_o & & & \\ & & a_n & a_{n-1} & \cdots & a_o & & \\ & & \cdots & \cdots & \cdots & \cdots & & \\ & & & a_n & a_{n-1} & \cdots & a_o & \\ b_m & b_{m-1} & \cdots & b_o & & & & \\ & b_m & b_{m-1} & \cdots & b_o & & & \\ & & b_m & b_{m-1} & \cdots & b_o & & \\ & & \cdots & \cdots & \cdots & \cdots & & \\ & & & b_m & b_{m-1} & \cdots & b_o & \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\\a\end{matrix}}\right\} m \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\\a\end{matrix}}\right\} n \text{ rows} \end{matrix}$$

where the remaining entries are all zero. A proof of this equivalence can be found in the classical reference on resultants, Chapter 12 of [U], but a more modern (and generalized) treatment can be found in [La].

Using the Sylvester matrix, we can also define the resultant of two multivariate complex polynomials, f and g, in the indeterminates $x, x_1, ..., x_k$, with respect to one of these indeterminates. If we treat f and g as polynomials in x whose coefficients are themselves complex polynomials in the remaining indeterminates $x_1, ..., x_k$, then the resultant of f and g, with respect to x, is defined to be the determinant of the Sylvester matrix given exactly as in the single variable case. In general the resultant of two multivariate polynomials will then be a polynomial in the remaining indeterminates $x_1, ...,$ x_k and not a single number as with single variable polynomials. Furthermore, if for some

$(z_0, z_1, ..., z_k) \in \mathbb{C}^{k+1}$ we have that both $f(z_0, z_1, ..., z_k) = 0$ and $g(z_0, z_1, ..., z_k) = 0$, then $R(f, g)(z_1, ..., z_k) = 0$ as well.

Thus, finding the resultant of two polynomials f and g in the indeterminates x, x_1, ..., x_k, provides a method by which we can eliminate the indeterminate x to produce a new polynomial in x_1, ..., x_k, namely $R(f, g)$, which maintains the important information about the common roots of f and g. This is exactly what we wish to do with the polynomials $p(M, t)$ and $q(L, M, t)$: eliminate t and produce a new polynomial, $R(p, q)$, that vanishes whenever both p and q simultaneously vanish.

The A-polynomial, by definition, is not divisible by L or M and has no overall integral factor. In addition, after we remove factors of the form $(L - 1)$, by [Bu], there will be only one type of irreducible factor left. As such, the A-polynomial A_{-n} of K_{-n} can be obtained by deleting factors of the form L, M, and $(L - 1)$, as well as any other repeated factors, from $R(p, q)$. Then any simultaneous solution of $p(M, t) = 0$ and $q(L, M, t) = 0$ will produce an irreducible representation ρ and a point $(L, M) \in \mathbb{C}^2$ for which

$$A_{-n}(L, M) = 0$$

Thus, we have shown that any solution to $p(M, t) = 0$ and $q(L, M, t) = 0$ corresponds to an irreducible representation of the knot group. Conversely, each irreducible representation gives a solution to $p(M, t) = 0$ and $q(L, M, t) = 0$. As a result, the polynomial A_{-n} is the defining polynomial for $\overline{\xi(R_i)}$, where R_i is the set of all irreducible representations of the knot group $\pi_1(K_{-n})$.

Suffice it to say that resultants are very difficult to calculate in general, which has been the main source of difficulty in our attempts to calculate in general the A-polynomial for these twist knots.

§3.0 Outline of Our Solution

In this section, we follow Section 2.3 above to develop the following program for calculating in general the A-polynomial of the twist knots K_{-n}, n a positive integer.

(1) Construct the matrices $X \underset{def}{=} \rho(x) = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix}$ and $Y \underset{def}{=} \rho(y) = \begin{pmatrix} M & 0 \\ t & M^{-1} \end{pmatrix}$.

(2) Construct the finite sequence of exponents (which we call the ε-array) for the word w of the knot group $\pi_1(K_{-n})$,

$$\varepsilon = \{\varepsilon_k\}_{k=1}^{4n} = \{(-1)^{\left\lfloor \frac{4n-1}{4n+1}k \right\rfloor}\}_{k=1}^{4n}$$

where $\lfloor \bullet \rfloor$ denotes the greatest integer function, and show that

$$\varepsilon = \{\{(-1)^{i-1}\}_{i=1}^{2n}, \{(-1)^{j}\}_{j=1}^{2n}\}.$$

(3) As a result, find in general $(YX^{-1})^n$ and $(Y^{-1}X)^n$, and then define the matrix

$$W \underset{def}{=} \rho(w) = Y^{\varepsilon_1} X^{\varepsilon_2} \dots Y^{\varepsilon_{4n-1}} X^{\varepsilon_{4n}} = (YX^{-1})^n (Y^{-1}X)^n.$$

(4) Similarly, find in general $(XY^{-1})^n$ and $(X^{-1}Y)^n$, and then define the matrix

$$W^* \underset{def}{=} \rho(w^*) = X^{\varepsilon_1} Y^{\varepsilon_2} \dots X^{\varepsilon_{4n-1}} Y^{\varepsilon_{4n}} = (XY^{-1})^n (X^{-1}Y)^n$$

(5) Construct the matrix P satisfying the group relator, and show that it has the form

$$P \underset{def}{=} XW - WY = \begin{pmatrix} 0 & (M^{-2})p(M,t) \\ (-M^{-2}t)p(M,t) & 0 \end{pmatrix}$$

where $p(M, t)$ is an integral polynomial in M and t we call the *p-polynomial*.

(6) Since the sum of the exponents in (2) is zero, construct the matrix Q as the projection of the longitude λ, and show that it has the form

$$Q = WW^* = \begin{pmatrix} (M^{-2})q'(M,t) & * \\ * & * \end{pmatrix}$$

where $q'(M, t)$ is an integral polynomial in M and t. We then define the q-*polynomial* to be $q(L, M, t) = M^2(q'(M,t)/M^2 - L) = q'(M,t) - M^2 L$

(7) Calculate the resultant of p and q, with respect to t.

§3.1 Finding a General Form for the ε-array

In the section, we will prove that the ε-array, i.e., the exponents on the word w of the knot group $\pi_1(K_{-n})$, has the general form

$$\underbrace{1, -1, 1, -1, ..., 1, -1, 1, -1,}_{2n \text{ terms}} \underbrace{-1, 1, -1, 1, ..., -1, 1, -1, 1}_{2n \text{ terms}}$$

and deduce from this that the sum of the exponents of the product ww^* is zero.

Proposition (3.1). Let $\varepsilon = \{\varepsilon_k\}_{k=1}^{4n} = \{(-1)^{\left|\frac{4n-1}{4n+1}k\right|}\}_{k=1}^{4n}$ for n a positive integer. Then $\varepsilon = \{\{(-1)^{i-1}\}_{i=1}^{2n}, \{(-1)^j\}_{j=1}^{2n}\}$.

Proof. First of all, as noted in Section 1.3 above, it is proven in [Bu] that the fundamental group for K_{-n} is symmetric in the sense that $\varepsilon_k = \varepsilon_{(4n+1)-k}$ for each $1 \leq k \leq 4n$. As such, it suffices to show that the first $2n$ terms of the ε-array are $\{\varepsilon_k\}_{k=1}^{2n} = \{(-1)^{k-1}\}_{k=1}^{2n}$.

Let k be an integer satisfying $1 \leq k \leq 2n$. Then

$$-1 < \frac{-4n}{4n+1} \leq \frac{-2k}{4n+1} \leq \frac{-2}{4n+1} < 0$$

so that

$$\left\lfloor \frac{4n-1}{4n+1}k \right\rfloor = \left\lfloor k - \frac{-2k}{4n+1} \right\rfloor = k-1$$

Thus, $\varepsilon_k = (-1)^{k-1}$ for each $k = 1, 2, \ldots, 2n$, Q.E.D.

§3.2 Finding a General Form for the *W*-Matrix

In the section, we will use the form of the ε-array given in Section 3.1 above to prove that under the representation ρ for the knot group $\pi_1(K_{-n})$ that was given in Section 2.3, the image of the word of this group, w, has the general form

$$W = \rho(w) = (YX^{-1})^n (Y^{-1}X)^n = \begin{pmatrix} a_{n-1}^2 + (M^2 t) b_n^2 & (M^{-1}) a_{n-1} b_n - M a_n b_n \\ (M^{-1} t) a_{n-1} b_n - (Mt) a_n b_n & a_n^2 + (M^{-2} t) b_n^2 \end{pmatrix}$$

where $X = \rho(x)$ and $Y = \rho(y)$ are the images of the generators of the group, that is

$$X = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix} \text{ and } Y = \begin{pmatrix} M & 0 \\ t & M^{-1} \end{pmatrix}$$

and (a_n) and (b_n) are sequences of polynomials (n a non-negative integer) given by

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n+k}{n-k} t^k, \text{ n} \geq 1, a_0 = 1, \text{ and } b_n = \sum_{k=0}^{n-1} (-1)^k \binom{n+k}{n-(k+1)} t^k, \text{ n} \geq 1, b_0 = 0$$

We begin by proving two important relationships between (a_n) and (b_n).

Lemma (3.2). $a_n = a_{n-1} - t\, b_n$ for all positive integers n.

Proof. Let n be a positive integer. Then notice that, by definition,

$$a_n + tb_n = \sum_{k=0}^{n} (-1)^k \binom{n+k}{n-k} t^k + \sum_{k=0}^{n-1} (-1)^k \binom{n+k}{n-(k+1)} t^{k+1}$$

$$= 1 + \sum_{k=1}^{n} (-1)^k \binom{n+k}{n-k} t^k + \sum_{k=1}^{n} (-1)^{k-1} \binom{n+(k-1)}{n-k} t^k$$

$$= 1 + \sum_{k=1}^{n} (-1)^k \left[\binom{n+k}{n-k} - \binom{n+(k-1)}{n-k} \right] t^k$$

$$= 1 + \sum_{k=1}^{n-1} (-1)^k \binom{(n-1)+k}{(n-1)-k} t^k = a_{n-1}$$

using Pascal's Formula. Thus, for each positive integer n, $a_n + t\, b_n = a_{n-1}$, Q.E.D.

Lemma (3.3). $b_n = b_{n-1} + a_{n-1}$ for all positive integers n.

Proof. Let n be a positive integer. Then notice that, by definition,

$$
\begin{aligned}
b_{n-1} + a_{n-1} &= \sum_{k=0}^{n-2}(-1)^k \binom{(n-1)+k}{(n-1)-(k+1)} t^k + \sum_{k=0}^{n-1}(-1)^k \binom{(n-1)+k}{(n-1)-k} t^k \\
&= \sum_{k=0}^{n-2}(-1)^k \binom{(n-1)+k}{(n-1)-(k+1)} t^k + \sum_{k=0}^{n-2}(-1)^k \binom{(n-1)+k}{(n-1)-k} t^k + (-1)^{n-1} t^{n-1} \\
&= \sum_{k=0}^{n-2}(-1)^k \left[\binom{(n-1)+k}{(n-1)-(k+1)} + \binom{(n-1)+k}{(n-1)-k} \right] t^k + (-1)^{n-1} t^{n-1} \\
&= \sum_{k=0}^{n-2}(-1)^k \binom{n+k}{(n-1)-k} t^k + (-1)^{n-1} t^{n-1} \\
&= \sum_{k=0}^{n-1}(-1)^k \binom{n+k}{n-(k+1)} t^k = b_n
\end{aligned}
$$

using Pascal's Formula. Thus, for each positive integer n, $b_n = b_{n-1} + a_{n-1}$, Q.E.D.

Now we find the general forms for the products $(YX^{-1})^n$ and $(Y^{-1}X)^n$.

Lemma (3.4). $(YX^{-1})^n = \begin{pmatrix} a_{n-1} & (-M)b_n \\ (M^{-1}t)b_n & a_n \end{pmatrix}$ for all positive integers n.

Proof. The proof will be by induction on n. The case for $n = 1$ is trivial, so suppose that the product $(YX^{-1})^n$ has the given form. Then

$$
\begin{aligned}
(YX^{-1})^{n+1} = (YX^{-1})^n (YX^{-1}) &= \begin{pmatrix} a_{n-1} & (-M)b_n \\ (M^{-1}t)b_n & a_n \end{pmatrix} \begin{pmatrix} 1 & -M \\ M^{-1}t & 1-t \end{pmatrix} \\
&= \begin{pmatrix} a_{n-1} - tb_n & (-M)(b_n + (a_{n-1} - tb_n)) \\ (M^{-1}t)(b_n + a_n) & a_n - t(a_n + b_n) \end{pmatrix} \\
&= \begin{pmatrix} a_n & (-M)(b_{n+1}) \\ (M^{-1}t)b_{n+1} & a_{n+1} \end{pmatrix}
\end{aligned}
$$

by Lemmas (3.2) and (3.3) above. Thus, $(YX^{-1})^{n+1}$ has the desired form as well, Q.E.D.

Lemma (3.5). $(Y^{-1}X)^n = \begin{pmatrix} a_{n-1} & (M^{-1})b_n \\ (-Mt)b_n & a_n \end{pmatrix}$ for all positive integers n.

Proof. The proof will be by induction on n. Again, the case for $n = 1$ is trivial, so suppose that the product $(Y^{-1}X)^n$ has the given form. Then, by definition,

$$(Y^{-1}X)^{n+1} = (Y^{-1}X)^n(Y^{-1}X) = \begin{pmatrix} a_{n-1} & (M^{-1})b_n \\ (-Mt)b_n & a_n \end{pmatrix}\begin{pmatrix} 1 & M^{-1} \\ -Mt & 1-t \end{pmatrix}$$

$$= \begin{pmatrix} a_{n-1} - tb_n & M^{-1}(b_n + (a_{n-1} - tb_n)) \\ (-Mt)(a_n + b_n) & a_n - t(a_n + b_n) \end{pmatrix}$$

$$= \begin{pmatrix} a_n & (M^{-1})b_{n+1} \\ (-Mt)b_{n+1} & a_{n+1} \end{pmatrix}$$

by Lemmas (3.2) and (3.3) above. Thus, $(Y^{-1}X)^{n+1}$ has the desired form as well, Q.E.D.

Using these two calculations, we can now find the general form of the W matrix for the twist knot K_{-n}:

Proposition (3.6). $W = \begin{pmatrix} a_{n-1}^2 + (M^2t)b_n^2 & (M^{-1})a_{n-1}b_n + (-M)a_nb_n \\ (M^{-1}t)a_{n-1}b_n + (-Mt)a_nb_n & a_n^2 + (M^{-2}t)b_n^2 \end{pmatrix}$

Proof. From Lemmas (3.4) and (3.5), it is immediate that

$$W = (YX^{-1})^n(Y^{-1}X)^n = \begin{pmatrix} a_{n-1} & (-M)b_n \\ (M^{-1}t)b_n & a_n \end{pmatrix}\begin{pmatrix} a_{n-1} & (M^{-1})b_n \\ (-Mt)b_n & a_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{n-1}^2 + (M^2t)b_n^2 & (M^{-1})a_{n-1}b_n + (-M)a_nb_n \\ (M^{-1}t)a_{n-1}b_n + (-Mt)a_nb_n & a_n^2 + (M^{-2}t)b_n^2 \end{pmatrix}$$

so that W has the desired form, Q.E.D.

§3.3 Finding a General Form for the *W**-Matrix

In the section, we will use the form of the ε-array given in Section 3.1 above to prove that under the representation ρ for the knot group $\pi_1(K_n)$ that was given in Section 2.3, the image of the reverse word w^* has the general form

$$W^* = \rho(w^*) = (XY^{-1})^n (X^{-1}Y)^n = \begin{pmatrix} a_n^2 + (M^2 t)b_n^2 & (M)a_{n-1}b_n - (M^{-1})a_n b_n \\ (Mt)a_{n-1}b_n - (M^{-1}t)a_n b_n & a_{n-1}^2 + (M^{-2}t)b_n^2 \end{pmatrix}$$

where $X = \rho(x)$ and $Y = \rho(y)$ are the images of the generators of the group, that is

$$X = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix} \text{ and } Y = \begin{pmatrix} M & 0 \\ t & M^{-1} \end{pmatrix}$$

and (a_n) and (b_n) are the sequences of polynomials given in Section 3.2 above.

We begin by finding the general forms for the products $(XY^{-1})^n$ and $(X^{-1}Y)^n$.

Lemma (3.7). $(XY^{-1})^n = \begin{pmatrix} a_n & (M)b_n \\ (-M^{-1}t)b_n & a_{n-1} \end{pmatrix}$ for all positive integers n.

Proof. The proof will be by induction on n. The case for $n = 1$ is trivial, so suppose that the product $(XY^{-1})^n$ has the given form. Then, by definition,

$$(XY^{-1})^{n+1} = (XY^{-1})^n (XY^{-1}) = \begin{pmatrix} a_n & (M)b_n \\ (-M^{-1}t)b_n & a_{n-1} \end{pmatrix} \begin{pmatrix} 1-t & M \\ -M^{-1}t & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_n - t(a_n + b_n) & M(a_n + b_n) \\ (-M^{-1}t)(b_n + (a_{n-1} - tb_n)) & a_{n-1} - tb_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{n+1} & (M)b_{n+1} \\ (-M^{-1}t)b_{n+1} & a_n \end{pmatrix}$$

by Lemmas (3.2) and (3.3) above. Thus, $(XY^{-1})^{n+1}$ has the desired form as well, Q.E.D.

Lemma (3.8). $(X^{-1}Y)^n = \begin{pmatrix} a_n & (-M^{-1})b_n \\ (Mt)b_n & a_{n-1} \end{pmatrix}$ for all positive integers n.

Proof. The proof will be by induction on n. Again, the case for $n = 1$ is trivial, so suppose that the product $(X^{-1}Y)^n$ has the given form. Then, by definition,

$$(X^{-1}Y)^{n+1} = (X^{-1}Y)^n(X^{-1}Y) = \begin{pmatrix} a_n & (-M^{-1})b_n \\ (Mt)b_n & a_{n-1} \end{pmatrix}\begin{pmatrix} 1-t & -M^{-1} \\ Mt & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_n - t(a_n + b_n) & (-M^{-1})(a_n + b_n) \\ (Mt)(b_n + (a_{n-1} - tb_n)) & a_{n-1} - tb_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{n+1} & (-M^{-1})b_{n+1} \\ (Mt)b_{n+1} & a_n \end{pmatrix}$$

by Lemmas (3.2) and (3.3) above. Thus, $(X^{-1}Y)^{n+1}$ has the desired form as well, Q.E.D.

Using these two calculations, we can now find the general form of the W^* matrix for the twist knot K_{-n}:

Proposition (3.9). $W^* = \begin{pmatrix} a_n^2 + (M^2t)b_n^2 & (M)a_{n-1}b_n - (M^{-1})a_nb_n \\ (Mt)a_{n-1}b_n - (M^{-1}t)a_nb_n & a_{n-1}^2 + (M^{-2}t)b_n^2 \end{pmatrix}$

Proof. From Lemmas (3.7) and (3.8), it is immediate that

$$W^* = (XY^{-1})^n(X^{-1}Y)^n = \begin{pmatrix} a_n & (M)b_n \\ (-M^{-1}t)b_n & a_{n-1} \end{pmatrix}\begin{pmatrix} a_n & (-M^{-1})b_n \\ (Mt)b_n & a_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} a_n^2 + (M^2t)b_n^2 & (M)a_{n-1}b_n - (M^{-1})a_nb_n \\ (Mt)a_{n-1}b_n - (M^{-1}t)a_nb_n & a_{n-1}^2 + (M^{-2}t)b_n^2 \end{pmatrix}$$

so that W^* has the desired form, Q.E.D.

§3.4 Finding a General Form for the *p*-polynomial

In the section, we will use the results of Section 3.2 above to show that the *p*-polynomial has the form

$$p(M,t) = (M^2 a_{n-1} + (M^2 - M^4 - 1)a_n)b_n + M^2 a_n^2$$

where (a_n) and (b_n) are the sequences of polynomials given in Section 3.2 above.

We begin with

Lemma (3.10). $P = XW - WY = \begin{pmatrix} 0 & (M^{-2})p(M,t) \\ (-M^{-2}t)p(M,t) & 0 \end{pmatrix}$ where p is the

polynomial $p(M,t) = (M^2-1)a_{n-1}b_n + (M^2 - M^4)a_n b_n + M^2 a_n^2 + t b_n^2$.

Proof. Notice that by Proposition 3.6,

$$XW = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix} \begin{pmatrix} (a_{n-1})^2 + (M^2 t)(b_n)^2 & (M^{-1})a_{n-1}b_n + (-M)a_n b_n \\ (M^{-1}t)a_{n-1}b_n + (-Mt)a_n b_n & (a_n)^2 + (M^{-2}t)(b_n)^2 \end{pmatrix}$$

$$= \begin{pmatrix} (M)a_{n-1}^2 + (M^3 t)b_n^2 + (M^{-1}t)a_{n-1}b_n + (-Mt)a_n b_n & a_{n-1}b_n + (-M^2)a_n b_n + a_n^2 + (M^{-2}t)b_n^2 \\ (M^{-2}t)a_{n-1}b_n + (-t)a_n b_n & (M^{-1})a_n^2 + (M^{-3}t)b_n^2 \end{pmatrix}$$

and

$$WY = \begin{pmatrix} (a_{n-1})^2 + (M^2 t)(b_n)^2 & (M^{-1})a_{n-1}b_n + (-M)a_n b_n \\ (M^{-1}t)a_{n-1}b_n + (-Mt)a_n b_n & (a_n)^2 + (M^{-2}t)(b_n)^2 \end{pmatrix} \begin{pmatrix} M & 0 \\ t & M^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} (M)a_{n-1}^2 + (M^3 t)b_n^2 + (M^{-1}t)a_{n-1}b_n + (-Mt)a_n b_n & (M^{-2})a_{n-1}b_n - a_n b_n \\ (t)a_{n-1}b_n + (-M^2 t)a_n b_n + (t)a_n^2 + (M^{-2}t^2)b_n^2 & (M^{-1})a_n^2 + (M^{-3}t)b_n^2 \end{pmatrix}$$

Thus,

$$XW - WY = \begin{pmatrix} 0 & (M^{-2})p(M,t) \\ (-M^{-2}t)p(M,t) & 0 \end{pmatrix}$$

with $p(M,t)=(M^2-1)a_{n-1}b_n+(M^2-M^4)a_nb_n+M^2a_n^2+tb_n^2$, Q.E.D.

Now we can simplify the form of the *p*-polynomial for the twist knot K_{-n}.

Proposition (3.11). $p(M,t)=(M^2a_{n-1}+(M^2-M^4-1)a_n)b_n+M^2a_n^2$

Proof. Starting with the form of *p* given in Lemma 3.10 above, we obtain, upon

applying Lemma 3.2,

$$\begin{aligned}
p(M,t)&=(M^2-1)a_{n-1}b_n+(M^2-M^4)a_nb_n+M^2a_n^2+tb_n^2\\
&=(M^2a_{n-1}-(a_{n-1}-tb_n))b_n+(M^2-M^4)a_nb_n+M^2a_n^2\\
&=(M^2a_{n-1}-a_n)b_n+(M^2-M^4)a_nb_n+M^2a_n^2\\
&=(M^2a_{n-1}+(M^2-M^4-1)a_n)b_n+M^2a_n^2
\end{aligned}$$

so that *p* has the desired form, Q.E.D.

§3.5 Finding a General Form for the *q*-polynomial

In the section, we will use the results of Sections 3.2 and 3.3 above to show that the *q*-polynomial $q(L, M, t) = q'(M, t) - M^2 L$ where $q'(M, t)$ has the form

$$q'(M, t) = M^2 a_{n-1}^2 a_n^2 + M^2 t(1 + M^2)(a_n^2 + a_{n-1}^2)b_n^2 - t(1 + M^4)a_{n-1}a_n b_n^2 + (M^6 t^2)b_n^4$$

and both (a_n) and (b_n) are the sequences of polynomials given in Section 3.2 above.

We prove this in

Proposition (3.12). $Q = WW^* = \begin{pmatrix} (M^{-2})q'(M,t) & * \\ * & * \end{pmatrix}$ where q' is the polynomial

$$q'(M, t) = M^2 a_{n-1}^2 a_n^2 + M^2 t(1 + M^2)(a_n^2 + a_{n-1}^2)b_n^2 - t(1 + M^4)a_{n-1}a_n b_n^2 + (M^6 t^2)b_n^4.$$

Proof. Notice that

$$WW^* = \begin{pmatrix} a_{n-1}^2 + (M^2 t)b_n^2 & (M^{-1})a_{n-1}b_n + (-M)a_n b_n \\ (M^{-1}t)a_{n-1}b_n + (-Mt)a_n b_n & a_n^2 + (M^{-2}t)b_n^2 \end{pmatrix}$$

$$\times \begin{pmatrix} a_n^2 + (M^2 t)b_n^2 & (M)a_{n-1}b_n - (M^{-1})a_n b_n \\ (Mt)a_{n-1}b_n - (M^{-1}t)a_n b_n & a_{n-1}^2 + (M^{-2}t)b_n^2 \end{pmatrix}$$

$$= \begin{pmatrix} a_{n-1}^2 a_n^2 + (M^2 t)a_{n-1}^2 b_n^2 + (M^2 t)a_n^2 b_n^2 + (M^4 t^2)b_n^4 & * \\ * & * \end{pmatrix}$$

$$+ \begin{pmatrix} (t)a_{n-1}^2 b_n^2 + (-M^{-2}t)a_{n-1}a_n b_n^2 + (-M^2 t)a_{n-1}a_n b_n^2 + (t)a_n^2 b_n^2 & * \\ * & * \end{pmatrix}$$

$$= \begin{pmatrix} (M^{-2})q'(M,t) & * \\ * & * \end{pmatrix}$$

with $q'(M, t) = M^2 a_{n-1}^2 a_n^2 + M^2 t(1 + M^2)(a_n^2 + a_{n-1}^2)b_n^2 - t(1 + M^4)a_{n-1}a_n b_n^2 + (M^6 t^2)b_n^4$,

Q.E.D.

§3.6 Observations about the A-polynomial

Unfortunately, even with the general forms of the *p*- and *q*-polynomials as given in Sections 3.4 and 3.5 above, we have been unable to find a general form for the A-polynomial since it requires the calculation of their resultant, i.e., the determinant of their very complicated Sylvester matrix. However, we have made many rather interesting observations about the A-polynomial for K_{-n}, which we summarize in this section.

In order to make it easier to visualize the A-polynomial (as well as to facilitate the extraction of something called the Newton polygon, which is discussed below), the authors of [CCGLS] suggest expressing the A-polynomial of a knot in terms of its coefficient matrix. That is, for the knot *K* we can write A-polynomial

$$A_K(L, M) = \sum_{i=0}^{m} \sum_{j=0}^{k} a_{ij} L^j M^{2i}$$

as

$$A_K(L, M) = \begin{bmatrix} 1 & M^2 & M^4 & \cdots & M^{2m} \end{bmatrix} \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0k} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1k} \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m0} & a_{m1} & a_{m2} & \cdots & a_{mk} \end{pmatrix} \begin{pmatrix} 1 \\ L \\ L^2 \\ \vdots \\ L^k \end{pmatrix}$$

where we will denote the coefficient matrix by $A(K) = (a_{ij})$ and call $A(K)$ the *A-matrix* of the knot *K*. Recall that the powers of *M* in $A(L, M)$ will all be even.

For the twist knots K_{-n}, with *n* a positive integer, we have made various observations about the form of the A-matrix. Some of these observations are catalogued below.

(1) The A-matrix will be $(4n + 1) \times (2n + 1)$ so that the A-polynomial $A_{-n}(L, M)$

will be

$$A_{-n}(L, M) = \begin{bmatrix} 1 & M^2 & M^4 & \cdots & M^{8n} \end{bmatrix} \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0,2n} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1,2n} \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2,2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{8n,0} & a_{8n,1} & a_{8n,2} & \cdots & a_{8n,2n} \end{pmatrix} \begin{pmatrix} 1 \\ L \\ L^2 \\ \vdots \\ L^{2n} \end{pmatrix}$$

The first few of these A-matrices are listed below.

$$A(K_{-1}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A(K_{-2}) = \begin{pmatrix} 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & 0 & 0 \\ -1 & -3 & -6 & -3 & -1 \\ 0 & 0 & -3 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & -1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix}$$

$$A(K_{-3}) = \begin{pmatrix} 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & -5 & 6 & -1 & 0 \\ 0 & 0 & -3 & 6 & 0 & 0 & 0 \\ 0 & 0 & 10 & 5 & -4 & 0 & 0 \\ 0 & 3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -5 & -12 & -10 & 0 & 0 & 0 \\ -1 & -4 & -10 & -20 & -10 & -4 & -1 \\ 0 & 0 & 0 & -10 & -12 & -5 & 0 \\ 0 & 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & -4 & 5 & 10 & 0 & 0 \\ 0 & 0 & 0 & 6 & -3 & 0 & 0 \\ 0 & -1 & 6 & -5 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$A(K_{-4}) = \begin{pmatrix}
0 & 0 & 0 & 0 & -1 & 3 & -3 & 1 & 0 \\
0 & 0 & 0 & 0 & 7 & -15 & 9 & -1 & 0 \\
0 & 0 & 0 & 4 & -19 & 16 & -1 & 0 & 0 \\
0 & 0 & 0 & -21 & 15 & 15 & -5 & 0 & 0 \\
0 & 0 & -6 & 24 & 9 & -4 & 0 & 0 & 0 \\
0 & 0 & 21 & 25 & 13 & -15 & 0 & 0 & 0 \\
0 & 4 & -3 & -8 & 11 & 0 & 0 & 0 & 0 \\
0 & -7 & -25 & -45 & -35 & 0 & 0 & 0 & 0 \\
-1 & -5 & -15 & -35 & -70 & -35 & -15 & -5 & -1 \\
0 & 0 & 0 & 0 & -35 & -45 & -25 & -7 & 0 \\
0 & 0 & 0 & 0 & 11 & -8 & -3 & 4 & 0 \\
0 & 0 & 0 & -15 & 13 & 25 & 21 & 0 & 0 \\
0 & 0 & 0 & -4 & 9 & 24 & -6 & 0 & 0 \\
0 & 0 & -5 & 15 & 15 & -21 & 0 & 0 & 0 \\
0 & 0 & -1 & 16 & -19 & 4 & 0 & 0 & 0 \\
0 & -1 & 9 & -15 & 7 & 0 & 0 & 0 & 0 \\
0 & 1 & -3 & 3 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$A(K_{-5}) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -9 & 28 & -30 & 12 & -1 & 0 \\
0 & 0 & 0 & 0 & -5 & 38 & -63 & 32 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 36 & -77 & 20 & 27 & -6 & 0 & 0 \\
0 & 0 & 0 & 10 & -84 & 39 & 54 & -14 & 0 & 0 & 0 \\
0 & 0 & 0 & -54 & 28 & 44 & 42 & -21 & 0 & 0 & 0 \\
0 & 0 & -10 & 64 & 81 & 6 & -21 & 0 & 0 & 0 & 0 \\
0 & 0 & 36 & 63 & 70 & 42 & -56 & 0 & 0 & 0 & 0 \\
0 & 5 & -8 & -42 & -42 & 42 & 0 & 0 & 0 & 0 & 0 \\
0 & -9 & -42 & -105 & -168 & -126 & 0 & 0 & 0 & 0 & 0 \\
-1 & -6 & -21 & -56 & -126 & -252 & -126 & -56 & -21 & -6 & -1 \\
0 & 0 & 0 & 0 & 0 & -126 & -168 & -105 & -42 & -9 & 0 \\
0 & 0 & 0 & 0 & 0 & 42 & -42 & -42 & -8 & 5 & 0 \\
0 & 0 & 0 & 0 & -56 & 42 & 70 & 63 & 36 & 0 & 0 \\
0 & 0 & 0 & 0 & -21 & 6 & 81 & 64 & -10 & 0 & 0 \\
0 & 0 & 0 & -21 & 42 & 44 & 28 & -54 & 0 & 0 & 0 \\
0 & 0 & 0 & -14 & 54 & 39 & -84 & 10 & 0 & 0 & 0 \\
0 & 0 & -6 & 27 & 20 & -77 & 36 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 32 & -63 & 38 & -5 & 0 & 0 & 0 & 0 \\
0 & -1 & 12 & -30 & 28 & -9 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

(2) As you can see, these A-matrices have a lot of nice symmetry, but they also become complicated very quickly. Some of the rows have obvious patterns, while others are not so obvious. Using various techniques such as the Newton Forward Difference Formula implemented in Appendix 4.2 below, we have been able to make the following conjectures about some of the rows in the A-matrix:

a. The non-zero elements of the top row will be

$$(-1)^{(n-1)-0}\binom{n-1}{0}, (-1)^{(n-1)-1}\binom{n-1}{1}, (-1)^{(n-1)-2}\binom{n-2}{2},\ldots,(-1)^{(n-1)-(n-1)}\binom{n-1}{n-1}$$

which corresponds to the polynomial

$$\sum_{i=0}^{n-1}\binom{n-1}{i}L^{n+i}(-1)^{(n-1)-i} = L^n\left[\sum_{i=0}^{n-1}\binom{n-1}{i}L^i(-1)^{(n-1)-i}\right] = L^n(L-1)^{n-1}$$

b. The non-zero elements of the second row from the top will be

$$(-1)^{n-0}\binom{n-1}{0}(2(n-0)-1),(-1)^{n-1}\binom{n-1}{1}(2(n-1)-1),\ldots,(-1)^{n-(n-1)}\binom{n-1}{n-1}(2(n-(n-1))-1)$$

which corresponds to the polynomial

$$M^2\left[\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^{n-i}(2(n-i)-1)L^{n+i}\right]$$

$$= M^2L^n\left[\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^{n-i}(2(n-i)-1)L^i\right]$$

$$= (1-L-2n)L^nM^2(L-1)^{n-2}$$

c. Based upon the above observations, one would be led to guess that the third row from the top should involve a summation with the factor

$$\left(an^2+bni+ci^2+dn+ei+f\right)\binom{n-1}{i}$$

for suitably chosen integer constants a, b, c, d, e, and f. Unfortunately, the third row is sufficiently more complicated than the previous two rows that it evades such analysis. It does not seem to follow the pattern suggested above. Similar remarks apply to most of the remaining rows, with a notable exception being the middle row.

d. The middle row of the A-matrix is better understood. All of its elements are non-zero and appear to be correspond to the polynomial

$$M^{4n}\left[\sum_{i=0}^{n}\binom{n+i}{i}L^i+\sum_{i=1}^{n}\binom{2n-i}{i}L^{n+i}\right]$$

Our original impetus for studying the A-polynomial came from results contained in Dr. Mattman's doctoral thesis, [Mat], in which he worked out the Newton polygons for the twist knots. In [CL 97], the authors prove that the A-matrix has "one's in the corners." (Cf. the five A-matrices listed above.) This means the Newton polygons (see [CCGLS] for a definition of the Newton polygon) are essentially a "mod. 2" version of the A-polynomial. More precisely, the A-polynomial modulo two determines the Newton polygon, although not conversely.

As such, since the direct analysis of the A-polynomials $A_{-n}(L, M)$ eluded us, we decided to try working with the $A_{-n}(L, M)$ (mod. 2). In other words, we reduced each coefficient modulo two in order to simplify significantly the calculations involved in finding the resultant of $p(M, t)$ (mod. 2) and $q(L, M, t)$ (mod. 2) for the twist knot K_{-n}. We immediately noticed several nice patterns. Specifically, if we denote by $p_{-n}(M, t)$ the p-

polynomial for the twist knot K_{-n}, n a positive integer, then the coefficients of t in p_{-n}, for

n a power of two, are periodic in an extremely predicable way. See Figure 22 below.

$$p_{-1}(M,\ t) = t^2\ M^2 + M^2 + (M^4 + M^2 + 1)\,t + (M^4 + 1)$$

$$p_{-2}(M,\ t) = M^2\ t^4 + (M^4 + M^2 + 1)\,t^3 + (M^4 + M^2 + 1)\,t^2 + (M^4 + 1)\,t + M^2$$

$$p_{-3}(M,\ t) =$$
$$M^2\ t^6 + (M^4 + M^2 + 1)\,t^5 + (M^4 + M^2 + 1)\,t^4 + (M^4 + 1)\,t^3 + M^2\ t + M^4 + M^2 + 1$$

$$p_{-4}(M,\ t) = M^2\ t^8 + (M^4 + M^2 + 1)\,t^7 +$$
$$(M^4 + M^2 + 1)\,t^6 + (M^4 + 1)\,t^5 + M^2\ t^4 + (M^4 + 1)\,t^3 + M^2$$

$$p_{-5}(M,\ t) = M^2\ t^{10} + (M^4 + M^2 + 1)\,t^9 + (M^4 + M^2 + 1)\,t^8 + (M^4 + 1)\,t^7 + M^2\ t^5 +$$
$$(M^4 + M^2 + 1)\,t^4 + (M^4 + 1)\,t^3 + M^2\ t^2 + (M^4 + M^2 + 1)\,t + M^4 + M^2 + 1$$

$$p_{-6}(M,\ t) = M^2\ t^{12} + (M^4 + M^2 + 1)\,t^{11} + (M^4 + M^2 + 1)\,t^{10} + (M^4 + 1)\,t^9 +$$
$$M^2\ t^8 + (M^4 + 1)\,t^7 + M^2\ t^3 + (M^4 + M^2 + 1)\,t^2 + (M^4 + 1)\,t + M^2$$

$$p_{-7}(M,\ t) = M^2\ t^{14} + (M^4 + M^2 + 1)\,t^{13} + (M^4 + M^2 + 1)\,t^{12} + (M^4 + 1)\,t^{11} +$$
$$M^2\ t^9 + (M^4 + M^2 + 1)\,t^8 + (M^4 + 1)\,t^7 + M^2\ t + M^4 + M^2 + 1$$

$$p_{-8}(M,\ t) = M^2\ t^{16} + (M^4 + M^2 + 1)\,t^{15} + (M^4 + M^2 + 1)\,t^{14} +$$
$$(M^4 + 1)\,t^{13} + M^2\ t^{12} + (M^4 + 1)\,t^{11} + M^2\ t^8 + (M^4 + 1)\,t^7 + M^2$$

Figure 22. p-polynomials for the first few twist knots with negative tangles number.

From the above data, one can quickly make a conjecture about the general form

for p_{-n}, n a power of two, which is surprisingly much simpler than the general form given

in Section 3.4 above. These patterns persist as far as we have calculated. See Figure 23

below.

$$p_{-16}(M,\ t) = M^2\ t^{32} + (M^4 + M^2 + 1)\,t^{31} + (M^4 + M^2 + 1)\,t^{30} + (M^4 + 1)\,t^{29} +$$
$$M^2\ t^{28} + (M^4 + 1)\,t^{27} + M^2\ t^{24} + (M^4 + 1)\,t^{23} + M^2\ t^{16} + (M^4 + 1)\,t^{15} + M^2$$

$$p_{-32}(M,\ t) = M^2\ t^{64} + (M^4 + M^2 + 1)\,t^{63} + (M^4 + M^2 + 1)\,t^{62} + (M^4 + 1)\,t^{61} + M^2\ t^{60} +$$
$$(M^4 + 1)\,t^{59} + M^2\ t^{56} + (M^4 + 1)\,t^{55} + M^2\ t^{48} + (M^4 + 1)\,t^{47} + M^2\ t^{32} + (M^4 + 1)\,t^{31} + M^2$$

$$p_{-64}(M,\ t) = M^2\ t^{128} + (M^4 + M^2 + 1)\,t^{127} + (M^4 + M^2 + 1)\,t^{126} +$$
$$(M^4 + 1)\,t^{125} + M^2\ t^{124} + (M^4 + 1)\,t^{123} + M^2\ t^{120} + (M^4 + 1)\,t^{119} + M^2\ t^{112} +$$
$$(M^4 + 1)\,t^{111} + M^2\ t^{96} + (M^4 + 1)\,t^{95} + M^2\ t^{64} + (M^4 + 1)\,t^{63} + M^2$$

Figure 23. p-polynomials for the new few negative powers of two.

Especially striking, however, is the simplicity of the q-polynomials of q_{-n}, for *n* a power of two. Notice that again the coefficients of *t* are periodic. See Figure 24 below.

$$q_{-1}(L, M, t) = (M^4 + M^2)t^3 + (M^6 + M^4 + M^2 + 1)t^2 + (M^4 + 1)t + LM^2 + M^2;$$

$$q_{-2}(L, M, t) = (M^4 + M^2)t^7 + (M^6 + M^4 + M^2 + 1)t^6 + (M^4 + 1)t^3 + LM^2 + M^2;$$

$$q_{-3}(L, M, t) = (M^4 + M^2)t^{11} + (M^6 + M^4 + M^2 + 1)t^{10} +$$
$$(M^4 + 1)t^5 + (M^4 + M^2)t^3 + (M^6 + M^4 + M^2 + 1)t^2 + (M^4 + 1)t + LM^2 + M^2;$$

$$q_{-4}(L, M, t) = (M^4 + M^2)t^{15} + (M^6 + M^4 + M^2 + 1)t^{14} + (M^4 + 1)t^7 + LM^2 + M^2;$$

$$q_{-5}(L, M, t) = (M^4 + M^2)t^{19} + (M^6 + M^4 + M^2 + 1)t^{18} + (M^4 + M^2)t^{11} + (M^6 + M^4 + M^2 + 1)t^{10} +$$
$$(M^4 + 1)t^9 + (M^4 + 1)t^5 + (M^4 + M^2)t^3 + (M^6 + M^4 + M^2 + 1)t^2 + (M^4 + 1)t + LM^2 + M^2;$$

$$q_{-6}(L, M, t) = (M^4 + M^2)t^{23} + (M^6 + M^4 + M^2 + 1)t^{22} +$$
$$(M^4 + 1)t^{11} + (M^4 + M^2)t^7 + (M^6 + M^4 + M^2 + 1)t^6 + (M^4 + 1)t^3 + LM^2 + M^2;$$

$$q_{-7}(L, M, t) = (M^4 + M^2)t^{27} + (M^6 + M^4 + M^2 + 1)t^{26} + (M^4 + M^2)t^{19} + (M^6 + M^4 + M^2 + 1)t^{18} +$$
$$(M^4 + 1)t^{13} + (M^4 + 1)t^9 + (M^4 + M^2)t^3 + (M^6 + M^4 + M^2 + 1)t^2 + (M^4 + 1)t + LM^2 + M^2;$$

$$q_{-8}(L, M, t) = (M^4 + M^2)t^{31} + (M^6 + M^4 + M^2 + 1)t^{30} + (M^4 + 1)t^{15} + LM^2 + M^2;$$

Figure 24. q-polynomials for the first few twist knots with negative tangles number.

Again, the simplicity and periodicity of the coefficients in the q-polynomials persists as far as we have been able to calculate. See Figure 25 below.

$$q_{-16}(L, M, t) = (M^4 + M^2)t^{63} + (M^6 + M^4 + M^2 + 1)t^{62} + (M^4 + 1)t^{31} + LM^2 + M^2;$$

$$q_{-32}(L, M, t) = (M^4 + M^2)t^{127} + (M^6 + M^4 + M^2 + 1)t^{126} + (M^4 + 1)t^{63} + LM^2 + M^2;$$

$$q_{-64}(L, M, t) = (M^4 + M^2)t^{255} + (M^6 + M^4 + M^2 + 1)t^{254} + (M^4 + 1)t^{127} + LM^2 + M^2;$$

Figure 25. q-polynomials for the new few negative powers of two.

We expect that these patterns in the p- and q-polynomials could be verified quite easily since, as proven in [G], reducing binomial coefficients modulo two is a very simple processes. Specifically, for positive integers n and m,

$$\binom{m}{n} (\text{mod. } 2) = m \text{ XOR } n$$

where XOR refers to the "exclusive or" operation being applied to the binary representations of n and m. See [G] for a more complete explanation.

Furthermore, we noticed that the A-matrix for K_{-2^n}, for n a positive integer, also follows a very nice pattern:

$$A(K_{-1}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} (\text{mod. } 2)$$

$$A(K_{-2}) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} (\text{mod. } 2)$$

$$A(K_{-4}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} (\text{mod. } 2)$$

In other words, the A-matrix $A(K_{-2^n})$ has a predictable arrangement of rows containing 2^k ones (k a non-negative integer not greater than n), although it's not very easy (or enlightening) to describe this pattern algebraically. Suffice it to say that this pattern persists as far as we've been able to calculate, to K_{-64}, whose A-matrix would require several pages to print. (Although it would be relatively easy to calculate the p- and q-polynomials for higher tangle number, we are constrained in calculating A-polynomials by computer time. The calculation of A_{-64} required several days on a fast computer.) Based on the nice patterns observed above for the p- and q-polynomials modulo two, we expect that this pattern persists for larger powers of two.

One final observation relates to the observed trivial form of the C-polynomial modulo two of K_{-n}, for n a positive integer. Given a knot K, the C-polynomial C_K is essentially a simplified form of the A-polynomial in a single variable, which we denote t. As such, it is not necessarily as powerful a knot invariant, but it still contains a lot of useful information about the knot K. A more complete definition and discussion of the C-polynomial can be found in [Z].

We denote by C_{-n} the C-polynomial of twist knot K_{-n}. Then, from our calculations, we have verified that $C_{-n} = t^{4n}$ (mod. 2) for $1 \leq n \leq 21$. Thus, although the A-polynomial modulo two still contains important information about K_{-n} (as shown by its Newton polygon), we conjecture that the C-polynomial modulo two reduces to this completely trivial form and so does not maintain any really important information about the knot.

§4.0 Description of Appendices

4.1 *Mathematica* Program for A-polynomials

Contains the source code we used for generating many A-polynomials for Twist Knots.

4.2 *Mathematica* Program for Difference Tables

Contains the source code we used for analyzing integer sequences when looking for patterns in the A-polynomials.

4.3 *Mathematica* Program for Resultant Matrices

Contains the source code we used for generating resultant matrices.

4.4 *Mathematica* Program for Subresultant Chain

Contains the source code we used for generating subresultant chains for the p- and q-polynomials.

4.5 *Mathematica* Program for C-polynomials

Contains the source code we used for generating many C-polynomials for twist knots.

§4.1 *Mathematica* **Program for A-polynomials**

Title of Program: A-polynomial Generator for Twist Knots

Author: Isaiah Lankham

Date Written: March 24, 2001

Program Version: 0.3 β (Initial Public Beta Release)

Description: As described in [CCGLS], the algorithm implemented calculates the A-polynomial for twist knots given the tangle number. It relies on taking the resultant of two polynomials that are generated by the representation of the fundamental group of the knot complement. The A-polynomial will then be the square-free determinant of the resultant matrix.

NB: This can take a *very* long time—even on a fast computer! Timing results are thus also included.

```
CalcApolynomial[ n_ ]  :=
(* n = the tangle number of the knot for which you wish to calculate the A-poly.
*)
Module[{ α, β, ε, i, X, X1, Y, Y1, W, Ws, P, p2,
     Q, q2, A},
(* Set α and β here. *)

   Print[ "What follows is the calculation of the p and q polynomial \
necessary for generating the A-polynomial for Twist Knot ", Subscript[ K, n],
     " :"];
   If[ IntegerQ[ n], If[ n >= 0, β = 4*n - 1, β = 4*(-n) + 1],
     Print[ "Error! n is not an integer!"]; Return[]];
 α = β - 2;

(* Declare ε - array and X - and Y - Matrices here. *)

Clear[ ε];
```

```
For[ i = 1, i <= β - 1, i++,
        ε [ i]  = (-1)^(Floor[ i* α / β] );
      ];
X = { {M, 1}, {0, 1/M} };
X1 = Inverse[ X];
Y = { {M, 0}, {t, 1/M} };
Y1 = Inverse[ Y];

(* Produce the W - matrix here. *)
Clear[ i];
For[ i = 1; W = {{ 1, 0}, {0, 1}}, i <= β - 1, i += 2,
        If[ ε [ i]  == 1, W = Factor[ W] .Y, W = Factor[ W] .Y1];
        If[ ε [ i + 1]  == 1, W = Factor[ W] .X, W = Factor[ W] .X1];
    ];

(* Produce the P - matrix and p - polynomial here. *)
P = X.W - W.Y;
p = Expand[ P[[ 1, 2]]];
p = M^-Exponent[ p, M, Min]*Factor[ p];
Print[ "\t The p-polynomial is given by"];
Print[ "\t p(t,M) = ", p];

(* Produce the Ws - matrix here. *)
Clear[ i];
For[ i = β - 1; Ws = {{ 1, 0}, {0, 1}}, i >= 2, i -= 2,
        If[ ε [ i]  == 1, Ws = Factor[ Ws] .X, Ws = Factor[ Ws] .X1];
        If[ ε [ i - 1]  == 1, Ws = Factor[ Ws] .Y, Ws = Factor[ Ws] .Y1];
    ];

(* Produce the Q - matrix and q - polynomial here. *)
Clear[ i];
For[ i = 1; signSum = 0, i <= β - 1, i++,
    signSum += ε [ i]];
Q = MatrixPower[ X, -2* signSum] .W.Ws;
q = Q[[ 1, 1]];
q = M^-Exponent[ q, M, Min]*Factor[ q - L];
Print[ "\t The q-polynomial is given by"];
Print[ "\t q(t,M,L) = ", q];
    (* Take the resultant and produce the A - polynomial here. *)

    Timing[ R = Factor[ Resultant[ Collect[ p, t], Collect[ q, t], t]];];

(* Set the A - polynomial here *)
A = Extract[ FactorList[ R], { 3, 1}];
Print[ "\t The square-free determinant of the resultant matrix is then the \
A-polynomial\n\n\t A(L,M) = "];
Print[ "\t", A];
```

```
    (* Calculate the A - polynomial matrix here. *)
Clear[ i];
Clear[ j];
A = Collect[ A, M];

T = Timing[ For[ i = 0; Amatrix = {}, i <= Exponent[ A, M], i += 2,
                currentCoef = Coefficient[ A, M, i];
                For[ j = 0; Arow = {}, j <= Exponent[ A, L], j++,
                       AppendTo[ Arow, Coefficient[ currentCoef, L, j]];
                ];
                AppendTo[ Amatrix, Arow];
        ];];
    Print[ "\t The Matrix form of the A-polynomial is then"];

    Print[ "\t", MatrixForm[ Amatrix]];

    Print[ "\t This calculation took a total of ", T[[ 1]]];

    Return[];
];
```

```
    (* Calculate the A - polynomial matrix here. *)
```

§4.2 *Mathematica* **Program for Difference Tables**

Title of Program: Newton Forward Difference Formula

Author: Isaiah Lankham

Date Written: July 11, 2001

Program Version: 0.2 β (Initial Public Beta Release)

Description: This program implements an algorithm for calculating the Newton Forward Difference Formula (NFDF) for a sequence given enough of the initial terms. The method is to construct a Difference Table of Order 1 until the differences become constant.

If the differences do not become constant, then the program exits. However, if the differences do become constant, then the NFDF is calculated.

```
(* Set List of first few sequence elements here and press Shift - Enter.
   An Example is entered below. *)

seqList = { 4 - 1, 56 - 32, 240 - 150, 680 - 440, 1540 - 1015, 3024 - 2016};

(* Code to calculate difference table. *)
nfdf = seqList[[ 1]];
Print[ "Difference Table:"];
For[j = 1; constantFlag = 0; initialLength = Length[ seqList]; coefList = {},
    j < initialLength && constantFlag == 0, j++,
      constantFlag = 1;
      For[ m = 1, m < Length[ seqList], m++,
           If[ seqList[[ m + 1]] != seqList[[ m]], constantFlag = 0];
          ];
      Print[ seqList];
      For[ i = 1; tempList = {}, i < Length[ seqList], i++,
           AppendTo[ tempList, seqList[[ i + 1]] - seqList[[ i]]];
          ];
      seqList = tempList;
```

```
              AppendTo[ coefList, seqList[[ 1]]];
                  ];


(* Code to calculate Newton Forward Difference Polynomial. *)
If[
  constantFlag == 0,
  Print[ "The Difference Table did not become constant. Try more terms or a \
model other than a finite polynomial."],
        Print[ "The Order 1 Difference Table of Depth ", j - 2,
      " was found to be constant."];
        Print[ "This gives the Newton Foward Difference Formula"];
        Clear[ k];
        For[ l = 1, l <= Length[ coefList], l++,
              nfdf += coefList[[ La]]*Binomial[ k, l];
            ];
        Print[ "\n      f(k) = ", TraditionalForm[ nfdf],
      ", \[ ForAll] k \[ Element] { 0,1,2...},"];
        Print[ "\nwhich simplified is f:{ 0,1,2,...} -> \[ DoubleStruckCapitalZ] by
\n\
\n      f(k)= ",
TraditionalForm[ Factor[ Simplify[ Expand[ nfdf]]]], " ."];
          ];


(* Code to predict the next few values. *)
If[ constantFlag == 1,
        f[ k_] = nfdf;
        For[ n = initialLength; predList = {}, n <= initialLength + 4, n++,
              AppendTo[ predList, f[ n]];
            ];
        Print[ "If this is correct, then the next few terms in the sequence are ",
      predList, "."];
        ];
```

§4.3 *Mathematica* **Program for Resultant Matrices**

Title of Program: Resultant Matrix Builder

Author: Isaiah Lankham

Date Written: March 24, 2001

Version: 0.3 β (Initial Public Beta Release)

Description: This program simply builds the Sylvester matrix for two polynomials, *f* and *g*, with respect to *x*, whose determinant is the resultant of *f* and *g*.

```
ResultantMatrix[ f_, g_, x_] :=
  Module[{ fDeg, fCoefList, gDeg, gCoefList, i, j, k, tempList},
    (* First build the coefficient lists for f and g. *)
    fDeg = Exponent[ f, x];
    gDeg = Exponent[ g, x];
    For[ i = fDeg; fCoefList = {}, i ≥ 0, i--,
      AppendTo[ fCoefList, Coefficient[ f, x, i]];];
    For[ i = gDeg; gCoefList = {}, i ≥ 0, i--,
      AppendTo[ gCoefList, Coefficient[ g, x, i]];];

    (* Now build the resultant matrix. *)
    ResMat = {};
    (* Build the top rows corresponding to f's coefficients. *)
    For[ i = 1; tempList = fCoefList, i ≤ gDeg, i++; tempList = fCoefList,
        For[ j = 0, j ≤ i - 2, j++, PrependTo[ tempList, 0];];
        For[ k = i + fDeg + 1, k ≤ fDeg + gDeg, k++, AppendTo[ tempList, 0];];
        AppendTo[ ResMat, tempList];
      ];

    (* Build the bottom rows corresponding to g's coefficients. *)
    For[ i = 1; tempList = gCoefList, i ≤ fDeg, i++; tempList = gCoefList,
        For[ j = 0, j ≤ i - 2, j++, PrependTo[ tempList, 0];];
        For[ k = i + gDeg + 1, k ≤ fDeg + gDeg, k++, AppendTo[ tempList, 0];];
        AppendTo[ ResMat, tempList];
      ];
    (* Finally, return the resulting matrix. *)
    Return[ ResMat];
    ];
```

§4.4 *Mathematica* Program for Subresultant Chain

Title of Program: Subresultant Chain Calculator

Author: Isaiah Lankham

Date Written: March 20, 2001

Version: 0.4 β (Initial Public Beta Release)

Description: This program implements all of the functions needed to calculate subresultant chains for two polynomials. The termination of this sequence is then the resultant of the two polynomials. The algorithm implemented below can be found in [Lo]. Timing results are printed out after each step along with the total time taken to get to that step.

```
(* Returns leading coefficient of polynomial A in indeterminate x. *)
LeadingCoefficient[ A_, x_] := Module[{}, Return[ Coefficient[ A, x, Exponent[ A,
x]]]];

(* Returns the sparse pseudo-remainder of polynomials A and B in x. *)
SparsePseudoRemainder[ A_, B_, x_] := Module[{bn, k, i, Rem, e},
     Rem[ 0] = A;
     i = Exponent[ A, x]  - Exponent[ B, x]  + 1;
     bn = LeadingCoefficient[ B, x];
     e = 0;
     Rem[ 1]  =
       PolynomialMod[
         Expand[ bn*Rem[ 0]  - x^(i - 1)*LeadingCoefficient[ Rem[ 0], x]*B], 2];
     For[ k = 2, k ≤ i, k++,
       If[ Exponent[ Rem[ k - 1], x]  < Exponent[ A, x]  - k + 1,
           (* Then *) Rem[ k]  = Rem[ k - 1]; e++,
           (* Else *)
           Rem[ k]  =
              Expand[ PolynomialMod[
                 bn*Expand[ Rem[ k - 1]]  -
                   x^(i - k)*LeadingCoefficient[ Expand[ Rem[ k - 1]], x]*B, 2]];
           ];
```

```
        ];

        Return[ Expand[ bn^e*Rem[ i]]];
        ];


(* Returns true if polynomial f is the zero polynomial. *)
ZeroPolynomialQ[ f_, x_] := Module[{},
        If[ Length[ CoefficientList[ f, x]] == 0,
            (* Then *) Return[ True],
            (* Else *) Return[ False],
            (* Default *) Print[ "Error!"]; Return[ Null];
            ];
        ];


(* Returns the subresultant chain of polynomials A and B in indeterminate x. *)
SubResChain[ A_, B_, x_] :=
  Module[{ m, n, i, j, k, steps, r, S, R, calctime, totaltime = 0},

    (* Step One *)
    m = Exponent[ A, x];
    n = Exponent[ B, x];

    If[ m > n, steps = j = m - 1, steps = j = n];

    calctime[ j + 1] = Timing[ S[ j + 1] = A][[ 1]];
    totaltime += calctime[ j + 1];
    Print[ "S[ ", j + 1, "]=", Collect[ Factor[ S[ j + 1]], x]];
    Print[ "\t Calculating S[ ", j + 1, "] took ", calctime[ j + 1]];
    Print[ "\t Total calculation time so far is ", totaltime];

calctime[ j] = Timing[ S[ j] = B][[ 1]];
totaltime += calctime[ j];
Print[ "S[ ", j, "]=", Collect[ Factor[ S[ j]], x]];
Print[ "\t Calculating S[ ", j, "] took ", calctime[ j]];
Print[ "\t Total calculation time so far is ", totaltime];

R[ j + 1] = 1;

While[ j ≥ 0,
  (* Step Two *)

  If[ ZeroPolynomialQ[ S[ j], x], r = -1, r = Exponent[ S[ j], x]];

  (* Step Three *)
  For[ k = j - 1, k ≥ r + 1, k--, S[ k] = 0;];
```

```
(* Step Four *)
If[ j > r && r ≥ 0,
    calctime[ r] =
      Timing[ S[ r] =
            Expand[ PolynomialMod[ (LeadingCoefficient[ S[ j], x])^(j - r)*
                  S[ j]/(R[ j + 1])^(j - r), 2]]][[ 1]];
    totaltime += calctime[ r];
    Print[ "S[ ", r, "] =", Collect[ S[ r], x]];
    Print[ "\t Calculating S[ ", r, "] took ", calctime[ r]];
    Print[ "\t Total calculation time so far is ", totaltime];
    ];

(* Step Five *)
If[ r ≤ 0,
    (*

      For[ i = steps, i ≥ 0, i--,
          Print[ "S[ ", i, "] =", Collect[ S[ i], t]];
          ];

      *)
    Return[ S[ 0]];
    ];

(* Step Six *)

        calctime[ r - 1] =
          Timing[ S[ r - 1] =
                PolynomialMod[
                  SparsePseudoRemainder[ Expand[ S[ j + 1]], Expand[ S[ j]], x]/
                    Expand[ (-R[ j + 1])^(j - r + 2)], 2]][[ 1]];
        totaltime += calctime[ r - 1];
        Print[ "S[ ", r - 1, "] =", Collect[ S[ r - 1], x]];
        Print[ "\t Calculating S[ ", r - 1, "] took ", calctime[ r - 1]];
        Print[ "\t Total calculation time so far is ", totaltime];
        j = r - 1;
        R[ j + 1] = LeadingCoefficient[ S[ j + 1], x];
        ];
    ];
```

§4.5 *Mathematica* **Program for C-polynomials**

Title of Program: C-polynomial Calculator

Author: Isaiah Lankham

Date Written: March 24, 2001.

Version: 0.1 β (Initial Public Beta Release)

Description: This program implements the definition of the C-polynomial given

in [Z] to generate the C-polynomial of a knot given the A-polynomial.

```
(* Returns the C-polynomial for the A-polynomial A. *)
CalcCpolynomial[ A] :=
  Module[{Apolynomial, i, j, k, l, m, TempValue, RootArray, RootCount,
     MultArray, PointArray, C},

    (* Calculate the C - polynomial factors here. *)

    Apolynomial[ L_, M_] = A;

    RootCount = 0;
    PointArray = {{-1, -1}, {-1, 1}, {1, -1}, {1, 1}};
    ValueArray = {Apolynomial[ -1, -1], Apolynomial[ -1, 1], Apolynomial[ 1, -1],
       Apolynomial[ 1, 1]};

    (* Create list of zeros of the A - polynomial. *)
    MultArray = {};
    For[ i = 1; RootArray = {}, i ≤ 4, i++,
       If[ ValueArray[[ i]] == 0, AppendTo[ RootArray, PointArray[[ i]]];
         RootCount++; AppendTo[ MultArray, 0]];
     ];

    (* Find Each Roots Multiplicity. *)
    For[ i = 1, i ≤ RootCount, i++,
       For[ TempValue = 0, TempValue == 0, MultArray[[ i]]++,
                   For[ j = MultArray[[ i]] + 1; k = 0, j ≥ 0, j--; k++,
                   PartialDer[ L_, M_] = D[ D[ Apolynomial[ L, M], {L, j}], {M,
k}];
```

```
                    If[ PartialDer[ RootArray[[i, 1]], RootArray[[i, 2]] ] ≠ 0,
                      TempValue += 1];
                              ];
                  ];
        ];

(* Calculate C - polynomial factors. *)
Print[ "\n\n"];
Print[ "\t For the given A-polynomial, we have the following factor list:"];

For[ j = 1; FactList = {}, j ≤ RootCount, j++, m = MultArray[[j]];
  AppendTo[ FactList,
    Sum[ Binomial[ m, i]/m!*
        ReplaceAll[
          ReplaceAll[ D[ D[ Apolynomial[ L, M], {M, m - i}], {L, i}],
            L -> RootArray[[j, 1]]], M -> RootArray[[j, 2]]]*t^i, {i, 0,
        m}]];
  Print[ "\t (", j, ") ", FactList[[j]]];
  ];

(* Create C - polynomial from Factors. *)
C[t] = 1;
For[ k = 1, k ≤ RootCount, k++,
    C[t] *= Expand[ FactList[[k]]];
  ];

Cpolynomial[ t_] = Expand[ Simplify[ Factor[ C[t]]]];

Print[
  "\t Thus, for the given A-polynomial, we have the following C-polynomial:"];
Print[ "\t", Expand[ Cpolynomial[ t]]];

Return[];
];
```

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