IHARA ZETA FUNCTIONS FOR GRAPHS OF RANK TWO.

ALEX RICHARDS

1. Introduction

Definition 1. *For a directed graph D, Let $\mathscr{L}_k(D)$ to be the set of subdigraphs of D with k vertices that consist of a union of disjoint cycles.*

Theorem 1.1. *For a graph G, the Ihara Zeta Function's reciprocal $\zeta_G(u)^{-1}$ is a polynomial with terms $c_k u^k$. Those coefficients are:*

$$c_k = \sum_{L \in \mathscr{L}_k(L^o G)} (-1)^{r(L)}$$

Proof. Using the findings of [SS], for a graph G it is known that $\zeta_G(u)^{-1} = \det(I - uT)$, where T is the adjacency matrix of the oriented line graph of G. The paper then found that in studying the characteristic polymonial of T,

$$\chi_T(u) = \det(T - uI) = u^{2m} + c_1 u^{2m-1} + ... + c_{2m}$$

one finds the following expression of the coefficients of the Ihara Zeta function's reciprocal:

$$\zeta_G(u)^{-1} = c_{2m} u^{2m} + c_{2m-1} u^{2m-1} + ... + c_1 u + 1.$$

Next, according to [SS, Lemma 12], the coefficients of this characteristic polynomial c_i are given by $(-1)^i$ times the sum of all $i \times i$ principal minors of T (labelled $\det(\tilde{T})$). Next, [SS, Lemma 13] states that given a digraph D, with linear subgraphs D_i for $i = 1, ..., n$, with D_i having e_i even cycles, then

$$\det(A) = \sum_{i=1}^{n} (-1)^{e_i}$$

Finally, in the proof of [SS, Theorem 7], they consider c_k for $2 \leq k < 2m$. Then the $k \times k$ principal minors of T can be considered as choosing k vertices of $L^o G$ and creating the subdigraph $\tilde{D}$ induced by them. $\tilde{D}$ will be an element of $\mathscr{S}_k(L^o G)$. Next, the prinicpal minors will be the determinants of the adjacency minor $\tilde{T}$ of $\tilde{D}$. Using a previous statement, for linear subgraphs $\tilde{D}_i$ for $i = 1, ..., j$, with $\tilde{D}_i$ having $e(\tilde{D}_i)$ even cycles, we have

$$\det(\tilde{T}) = \sum_{\tilde{D}_i \subseteq \tilde{D}} (-1)^{e(\tilde{D}_i)}$$

Date: May 2023.

We now diverge from the paper to offer an alternate simplification of the zeta coefficients, and summing this over all principal minors:

$$\begin{aligned} c_k &= \sum_{\tilde{D}\in\mathscr{S}_k(L^oG)} (-1)^k \sum_{\tilde{D}_i\subseteq\tilde{D}} (-1)^{e(\tilde{D}_i)} \\ &= \sum_{\tilde{D}\in\mathscr{S}_k(L^oG)} \sum_{\tilde{D}_i\subseteq\tilde{D}} (-1)^k(-1)^{e(\tilde{D}_i)} \end{aligned}$$

As each $\tilde{D} \in \mathscr{S}_k(L^oG)$ will be formed from a unique set of k vertices in L^oG, the set of all $\tilde{D}_i$ in the above double sum will be exactly $\mathscr{L}_k(L^oG)$ defined before, so we can simplify: [check argument validity]

$$\begin{aligned} c_k &= \sum_{L\in\mathscr{L}_k(L^oG)} (-1)^k(-1)^{e(L)} \\ &= \sum_{L\in\mathscr{L}_k(L^oG)} (-1)^{k\cdot e(L)} \end{aligned}$$

Let $L \in \mathscr{L}_k(L^oG)$ and assume k is odd. Then k is the sum of vertices in all cycles, so there must be an odd number of odd cycles. If the number of cycles $r(L)$ is even, then there are an odd number of even cycles, and if it is odd, then there are an even number of even cycles; the parity of $r(L)$ will be the opposite of $e(L)$. Similarly for k even, there are an even number of odd cycles, so the parity of the number of even cycles $e(L)$ will match the parity of $r(L)$. To summarize:

k	$e(L)$	$r(L)$	$k\cdot e(L)$
Even	Even	Even	Even
Even	Odd	Odd	Odd
Odd	Even	Odd	Odd
Odd	Odd	Even	Even

As $r(L)$ and $k \cdot e(L)$ have the same parity in all cases, we arrive at

$$c_k = \sum_{L\in\mathscr{L}_k(L^oG)} (-1)^{r(L)}$$

□

2. Families of rank 2 graphs

Definition 2. *A $G_{m,n}$ graph, with $m, n > 2$, given by $G_{m,n} = C_m \dot{\cup} C_n$, is the union of two cycles made by identifying one vertex of each. The order and size are $|G_{m,n}| = m + n - 1$ and $\|G_{m,n}\| = m + n$.*

Theorem 2.1. *For $3 \le m \le n$,*

$$\zeta_{G_{m,n}}(u)^{-1} = -3u^{2(m+n)} + 2u^{m+2n} + 2u^{2m+n} + u^{2n} + u^{2m} - 2u^n - 2u^m + 1.$$

Remark 1. *When $m = n$, this factors:*

$$\zeta_{G_{m,m}}(u)^{-1} = -(3u^m - 1)(u^{2m} - 1)(u^m - 1).$$

Following [SS], we first determine the structure of the oriented line graph, $L^oG_{m,n}$.

Lemma 2.2. *The oriented line graph $L = L^oG_{m,n}$ consists of a cube with four 'wings'. Denote the vertices of the cube $v_{a,b,c}$ with $(a, b, c) \in \{0, 1\}^3$ so that the (a, b, c) are the corresponding points on a cube in $\mathbb{R}^3$. Orient edges of the cube so that $v_{a,b,c}$ is a source (outdegree 3) if (a, b, c) has an even number of 1's and a sink (indegree 3) otherwise. The four wings are oriented cycles each using one of the vertical edges of the cube. The wings alternate C_m and C_n cycles as we go through the four vertical edges of the cube.*

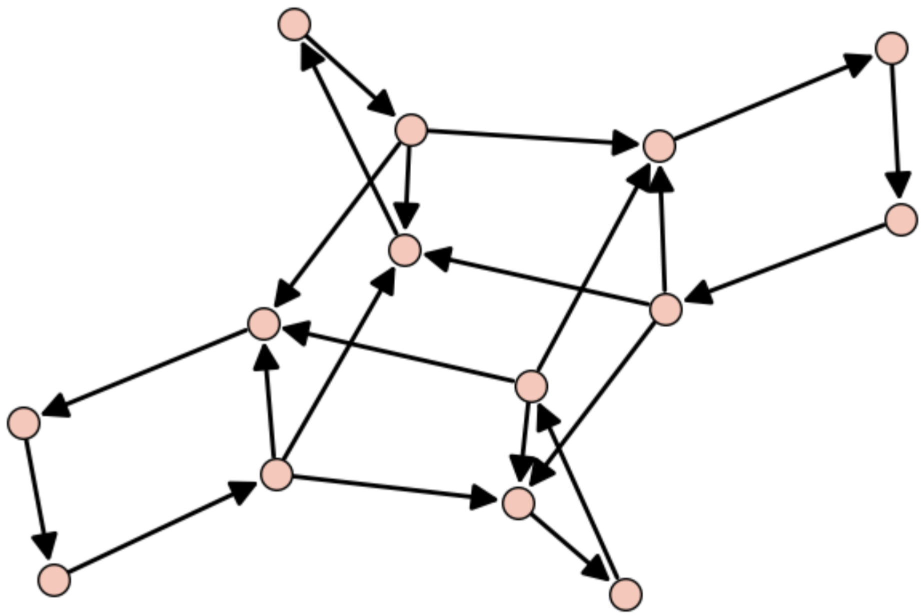

FIGURE 1. The oriented line graph $L^oG_{3,4}$

Proof. In $G_{m,n}$, let x denote the common vertex of the two cycles and $v_2, \ldots, v_m$ and $w_2, \ldots, w_n$ the remaining vertices of each cycle, in order as we traverse the cycles. Then $N(x) = \{v_2, v_m, w_2, w_m\}$ and the vertices of the cube in $L^oG_{m,n}$ are the eight directed edges incident on x in the symmetric digraph $D(G_{m,n})$. The four edges that terminate at x are the four sources in the cube and those that initiate at x are sinks.

The remaining edges of C_m and C_n generate the wings. Each of the two directed cycles around C_m in $D(G_{m,n})$ generates one wing in L and similarly for C_n. □

Proof. (of Theorem) To prove the theorem we need to study the linear subgraphs of $L = L^oG_{m,n}$. Since each vertex of the cube is either a source or a sink, in a linear subgraph each cube vertex has at most one edge from the cube. Then it must have exactly one edge from the cube and one edge from a wing. It follows that a cycle in a linear subgraph is either a wing, a C_{m+n} cycle using top and bottom edges of a cube face, or else a $C_{2(m+n)}$ cycle that uses all four wings.

As in [SS], we define $\mathcal{S}_k(L)$ as the set of order k subgraphs. For $\tilde{D} \in \mathcal{S}_k(L)$, $\mathcal{E}_k(\tilde{D})$ (respectively, $\mathcal{O}_k(\tilde{D})$) is the count of linear subgraphs with an even (resp., odd) number of even cycles.

Given our listing of the cycles of L above, the subgraphs that admit a linear subgraph have order $k \in \{2(m+n), m+2n, 2m+n, 2m, 2n, m, n\}$. For $k = 2(m+n)$, there are four ways to make a linear subgraph: A single $2(m+n)$ cycle, two $m+n$ cycles, an $m+n$ cycle with two wings (one a C_m, the other a C_n), or else by using all four wings.

Type	Count	$\mathcal{E}$	$\mathcal{O}$
$2(m+n)$ cycle	2	0	1
Two $m+n$ cycles	2	1	0
$m+n$ cycle and two wings	4	0	1
four wings	1	1	0

TABLE 1. Linear subgraphs for $k = 2(m+n)$

Table 1 list the number of ways each linear subgraph arises and the corresponding $\mathcal{E} = \mathcal{E}(\tilde{D})$ and $\mathcal{O} = \mathcal{O}(\tilde{D})$. Using [SS, Theorem 7],

$$\begin{aligned} c_{2(m+n)} &= \sum (-1)^{2(m+n)}(\mathcal{E} - \mathcal{O}) \\ &= 2(0-1) + 2(1-0) + 4(0-1) + (1-0) = -3 \end{aligned}$$

Type	Count	$\mathcal{E}$	$\mathcal{O}$
$m+n$ cycle and n wing	4	1	0
three wings	2	0	1

TABLE 2. Linear subgraphs for $k = m+2n$, m even

For $k = m + 2n$, there are two cases depending on the parity of m. Table 2 illustrates the situation when m is even. Then,

$$\begin{aligned} c_{m+2n} &= \sum (-1)^{m+2n}(\mathcal{E} - \mathcal{O}) \\ &= 4(1-0) + 2(0-1) = 2 \end{aligned}$$

When m is odd, the calculation is similar,

$$c_{m+2n} = -1\big(4(0-1) + 2(1-0)\big) = 2$$

and determining that $c_{n+2m} = 2$ is analogous.

A linear subgraph of order $2m$ necessarily consists of the two C_m wings. There is only one such linear subgraph and it has $\mathcal{E} = 1$, $\mathcal{O} = 0$, so that $c_{2m} = (1-0) = 1$. Similarly $c_{2n} = 1$.

A linear subgraph of order m is a single wing, but there are two ways to choose the wing. If m is even, $\mathcal{E} = 0$, $\mathcal{O} = 1$ and $c_m = 2(0-1) = -2$. When m is odd, we have instead $c_m = -1(2(1-0)) = -2$. Similarly, $c_n = -2$, whatever the parity of n. Finally, recall that the constant coefficient for $\zeta_G(u)^{-1}$ is 1. □

Definition 3. *A $G_{m,n,p}$ graph, with $m, n > \max\{p+1, 2\}$ and $p \geq 0$, is $G_{m,n,p} = C_m \overset{p}{\cup} C_n$, the union of two cycles made by identifying p consecutive edges of each (if $p = 0$, identify one vertex of each). The order and size are $|G_{m,n,p}| = m+n-p-1$ and $\|G_{m,n,p}\| = m+n-p$.*

Possible cycles (Figure 2)

- m - m wing (2 kinds)
- n - n wing (2 kinds)
- $2m$ - two opposite wings (unique)
- $2n$ - two opposite wings (unique)
- $m+n$ - two adjacent wings not sharing the pinch point (2 kinds)
- $m+n-2$ - 'sides' not passing through pinch points (2 kinds)
- $2m+n-2$ - one 'side' with opposite m wing (2 kinds)
- $m+2n-2$ - one 'side' with opposite n wing (2 kinds)
- $2m+2n-4$ - both 'sides' (unique)
- $2m+2n-2$ - all vertices (2 kinds), one 'side' with both opposite wings (2 kinds)

Possible cycles (Figure 3)

- m - m wing (2 kinds)
- n - n wing (2 kinds)
- $2m$ - two opposite wings (unique)
- $2n$ - two opposite wings (unique)
- $m+n$ - two adjacent wings not sharing a p vertical path (2 kinds)
- $m+n-2p$ - 'sides' not passing through pinch points (2 kinds)
- $2m+n-2p$ - one 'side' with opposite m wing (2 kinds)
- $m+2n-2p$ - one 'side' with opposite n wing (2 kinds)
- $2m+2n-4p$ - both 'sides' (unique)
- $2m+2n-2p$ - all vertices (2 kinds), one 'side' with both opposite wings (2 kinds)

$$\zeta_{G_{m,n,1}}(u)^{-1} = -4u^{2m+2n-2} + u^{2m+2n-4} + 2u^{m+2n-2} + 2u^{2m+n-2} \\ + u^{2n} + u^{2m} + 2u^{m+n} - 2u^{m+n-2} - 2u^{n} - 2u^{m} + 1$$

$$\zeta_{G_{m,n,p}}(u)^{-1} = -4u^{2m+2n-2p} + u^{2m+2n-4p} + 2u^{m+2n-2p} + 2u^{2m+n-2p} \\ + u^{2n} + u^{2m} + 2u^{m+n} - 2u^{m+n-2p} - 2u^{n} - 2u^{m} + 1$$

Remark 2. *This reproduces* [SS, Corollary 19] *when* $p = 1$.

Remark 3. *When* $m = n$, $u^{2m+2p} - 1$ *is a factor of* $\zeta_{G_{m,n,p}}(u)^{-1}$.

Definition 4. *An* $H_{m,n,l}$ *handcuff graph, with* $m, n > 2$ *and* $l \geq 0$, *is* C_m *connected to* C_n *by a path of* l *edges (if* $l = 0$, *join the two at a vertex). The order and size are* $|H_{m,n,l}| = m + n + l - 1$ *and* $\|H_{m,n,l}\| = m + n + l$.

Remark 4. *Observe that we could instead have defined negative values of* p *in* $G_{m,n,p}$ *graphs to represent these graphs (or negative values of* l *to represent* $G_{m,n,p}$ *graphs), but as the resulting Ihara Zeta Function formulas differ we have chosen not to.*

Possible cycles (Figure 4)

- m - m wing (2 kinds)
- n - n wing (2 kinds)
- $2m$ - two adjacent wings (unique)

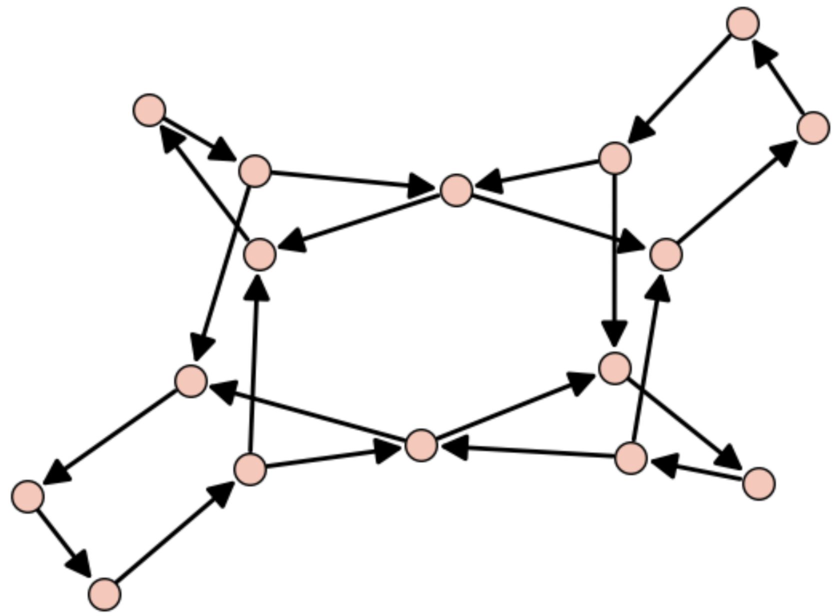

FIGURE 2. The oriented line graph $L^oG_{4,5,1}$

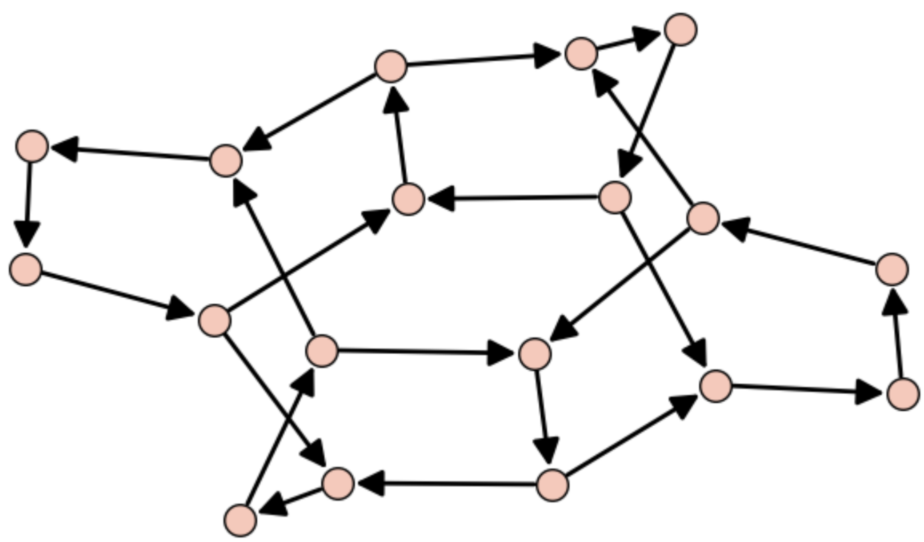

FIGURE 3. The oriented line graph $L^oG_{5,6,2}$

- $2n$ - two adjacent wings (unique)
- $m + n$ - two opposite wings (4 kinds)
- $2m + n$ - two m wings and one n wing (2 kinds)
- $m + 2n$ - one m wing and two n wings (2 kinds)
- $2m + 2n$ - all four wings (unique)

- $m + n + 2l$ - 'sides' navigating around one wing, through the center, to an opposite wing and back (4 kinds)
- $2m + n + 2l$ - one 'side' with opposite m wing (4 kinds)
- $m + 2n + 2l$ - one 'side' with opposite n wing (4 kinds)
- $2m + 2n + 2l$ - one 'side' with both opposite wings (4 kinds)

Formula for $H_{m,n,l}$ handcuff graph:

$$\begin{aligned}\zeta_{H_{m,n,l}}(u)^{-1} = &-4u^{2m+2n+2l} + u^{2m+2n} + 4u^{2m+n+2l} + 4u^{m+2n+2l} - 2u^{2m+n} - 2u^{m+2n} \\ &- 4u^{m+n+2l} + 4u^{m+n} + u^{2n} + u^{2m} - 2u^n - 2u^m + 1\end{aligned}$$

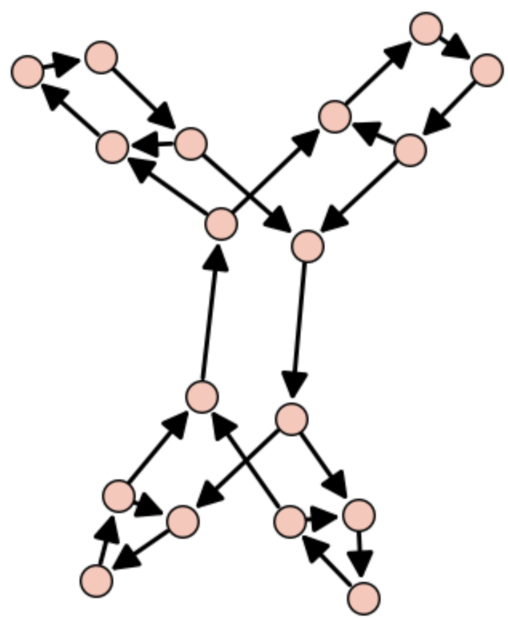

FIGURE 4. The oriented line graph $L^o H_{4,3,2}$

Formula for $G_{2,n}$ graph (union of bigon and C_n over a vertex): (Compare with zeta for $G_{m,n}$.)

Theorem 2.3. *For $3 \leq n$,*

$$\zeta_{G_{2,n}}(u)^{-1} = -3u^{2(2+n)} + 2u^{2+2n} + 2u^{4+n} + u^{2n} + u^4 - 2u^n - 2u^2 + 1.$$

And for $G_{1,n}$ graph (add a loop at one vertex of a C_n): (Again generalizes formula for $G_{m,n}$)

Theorem 2.4. *For $3 \leq n$,*

$$\zeta_{G_{1,n}}(u)^{-1} = -3u^{2(1+n)} + 2u^{1+2n} + 2u^{2+n} + u^{2n} + u^2 - 2u^n - 2u + 1.$$

Also, the rank two graphs that have only loops and multiedges agree with the expected specialization of the formula for $G_{m,n}$. Specifically, let B_2 denote a bouquet of two loops, BL be a bigon with a loop and BB be two bigons joined at a single vertex. Then

$$\zeta_{B_2}(u)^{-1} = \zeta_{G_{1,1}}(u)^{-1} = -3u^4 + 4u^3 + 2u^2 - 4u + 1$$
$$\zeta_{BL}(u)^{-1} = \zeta_{G_{1,2}}(u)^{-1} = -3u^6 + 2u^5 + 3u^4 - u^2 - 2u + 1$$
$$\zeta_{BB}(u)^{-1} = \zeta_{G_{2,2}}(u)^{-1} = -3u^8 + 4u^6 + 2u^4 - 4u^2 + 1$$

Let $G(m, 2, 1)$ denote a graph that is a C_m with one doubled edge. The zeta function is a specialization of that for $G(m, n, p)$:

$$\zeta_{G_{m,2,1}}(u)^{-1} = -4u^{2m+2} + 4u^{2m} + 4u^{m+2} + u^4 - 4u^m - 2u^2 + 1$$

In particular, for the theta graph $G_{2,2,1}$ we have

$$\zeta_{G_{2,2,1}}(u)^{-1} = -4u^6 + 9u^4 - 6u^2 + 1$$

Let $H_{2,n,l}$ denote a Handcuff graph where the cycle C_n is joined to a bigon by a path of l edges. The polynomial for $H_{2,n,l}$ is a special case of that for $H_{m,n,l}$ by substituting $m = 2$.

$$\begin{aligned}\zeta_{H_{2,n,l}}(u)^{-1} = &-4u^{2(2+n+l)} + u^{2(2+n)} + 4u^{2+2n+2l} + 4u^{4+n+2l} - 2u^{4+n} - 2u^{2+2n} \\ &- 4u^{2+n+2l} + u^{2n} + 4u^{2+n} + u^4 - 2u^n - 2u^2 + 1\end{aligned}$$

Let $H_{1,n,l}$ denote a Handcuff graph where the cycle C_n is connected to a path of l edges with a loop at its other end. The polynomial for $H_{1,n,l}$ is a special case of that for $H_{m,n,l}$, by substituting $m = 1$.

$$\begin{aligned}\zeta_{H_{1,n,l}}(u)^{-1} = &-4u^{2(1+n+l)} + u^{2(1+n)} + 4u^{1+2n+2l} + 4u^{2+n+2l} - 2u^{2+n} - 2u^{1+2n} \\ &- 4u^{1+n+2l} + u^{2n} + 4u^{1+n} + u^2 - 2u^n - 2u + 1\end{aligned}$$

There remain three related types of graph where we have a path of length l with a loop or a bigon at both ends. Let $H_{2,2,l}$ denote two bigons joined by a path of length l. Again, the polynomial specializes that for $H_{m,n,l}$.

$$\zeta_{H_{2,2,l}}(u)^{-1} = -4u^{2(4+l)} + u^8 + 8u^{6+2l} - 4u^{4+2l} - 4u^6 + 6u^4 - 4u^2 + 1$$

If we have a bigon and a loop joined by a path of length l, we'll write $H_{1,2,l}$. As usual, the polynomial specializes $H_{m,n,l}$.

$$\zeta_{H_{1,2,l}}(u)^{-1} = -4u^{2(3+l)} + u^6 + 4u^{4+2l} + 4u^{5+2l} - 2u^5 - u^4 - 4u^{3+2l} + 4u^3 - u^2 - 2u + 1$$

Finally, if $H_{1,1,l}$ denotes two loops joined by a path of length l, we again specialize $H_{m,n,l}$.

$$\zeta_{H_{1,1,l}}(u)^{-1} = -4u^{2(2+l)} + u^4 + 8u^{3+2l} - 4u^3 - 4u^{2+2l} + 6u^2 - 4u + 1$$

3. Generalized theta graphs

The term generalized theta graph has been used in diverse ways in the literature. Let $G = T(l_1, l_2, ..., l_n)$ denote a graph consisting of n internally disjoint paths of l_1, l_2, ..., l_n edges, respectively, between the distinct vertices v_0 and v_1. We will say that G is a *generalized theta graph.* In case $n = 3$, we refer to G simply as a theta graph. For example, the $G_{m,n,p}$ graphs of Definition 3 are $T(p, m-p, n-p)$ theta graphs.

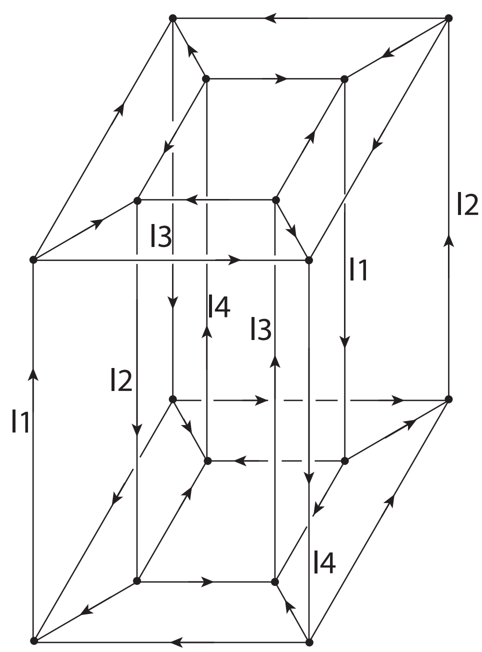

FIGURE 5. The oriented line graph $L^oT(l_1, l_2, l_3, l_4)$

Assuming $l_i \geq 1$ for $i = 1, 2, 3, 4$, let's find $\zeta_{T(l_1,l_2,l_3,l_4)}(u)^{-1}$. For a first pass, assume that at most one $l_i = 1$, so that we have no double edges. Figure 5 shows the oriented line graph. The vertical edges are labelled with l_i's which is the number of vertices on those edges.

4. Spanning Trees

Let κ_G denote the number of spanning trees of graph G. The following theorem is given as an exercise in [T].

Theorem 4.1. *The Ihara zeta function satisfies*

$$\frac{d^r}{du^r}\zeta_G(u)^{-1}\bigg|_{u=1} = (-1)^{r-1}2^r r!(r-1)\kappa_G,$$

where $r = |E| - |V| + 1$ *is the graph's rank.*

Making use of this formula, we determine κ_G for the graphs of rank two. If $r = 2$, the Theorem states

$$\kappa_G = -\frac{1}{8}\frac{d^r}{du^r}\zeta_G(u)^{-1}\Big|_{u=1}$$

It's easy to see that the $\kappa_{G_{m,n}} = mn$ since a spanning tree is formed by removing one edge of the m-cycle and one edge of the n cycle. This agrees with the result of Theorem 4.1 using the formula for $\zeta_{G_{m,n}}(u)^{-1}$ given above.

Similarly, $\kappa_{H_{m,n,l}} = mn$, which agrees with the result of Theorem 4.1 using the formula for $\zeta_{H_{m,n,l}}(u)^{-1}$ given above.

The graph $G_{m,n,p}$ is a theta graph: a union of three internally disjoint paths of length $m-p$, $n-p$, p between the distinct vertices v_0 and v_1. A spanning tree is formed by removing one edge from two of the three paths. Thus,

$$\begin{aligned}\kappa_{G_{m,n,p}} &= p(m-p) + p(n-p) + (m-p)(n-p)\\ &= mn - p^2.\end{aligned}$$

This agrees with the result of applying Theorem 4.1 to the equation for $\zeta_{G_{m,n,p}}(u)^{-1}$ given above.

5. Rank 2 graphs with equal zeta function

In this section we show that if G_1 and G_2 are rank 2 graphs that share the same Ihara Zeta function, then G_1 and G_2 are isomorphic.

For G of rank 2, the leading coefficient of $\zeta_G(u)^{-1}$ is $c_{2\|G\|} = -3$ or -4. This is in agreement with the formula of Kotani and Sunada [KS]:

$$c_{2|E|} = (-1)^{|E|-|V|}\prod_{v_i\in V}(d(v_i)-1).$$

In order for two rank 2 graphs to have the same leading term, they would have to have the same size and either both have a single degree 4 vertex, or both have a pair of degree 3 vertices. (Degree 2 vertices correspond to multiplying by 1 in the product.)

Theorem 4.1 implies the two graphs would have the same number of spanning trees and, as discussed in [SS], if both are simple, they must also have the same girth. We generalize the definition of girth to multigraphs by saying that a graph with a loop has girth one. If a multigraph has no loops, but does have bigons, we'll say the girth is two. Corollary 14 of [SS] shows that two multigraphs with the same Ihara zeta function must have the same girth. This implies that a simple graph cannot share its Ihara zeta function with a graph that has loops or bigons.

Suppose two graphs G_1 and G_2 of rank 2 share the same Ihara zeta function, both having leading coefficient -3 for $\zeta_G(u)^{-1}$. If both are simple, then they are both $G_{m,n}$ graphs. As G_1 and G_2 have the same order and girth, they must be isomorphic.

The non-simple rank 2 graphs with leading coefficient -3 are graphs of the form $G_{2,n}$ or $G_{1,n}$. Suppose G_1 is not simple and has an Ihara zeta function so that $\zeta_{G_1}(u)^{-1}$ has leading coefficient -3. If G_2 has the same Ihara zeta function, then, since they must have the same girth, G_2 is also not simple.

If G_1 has girth one, it has a loop and G_2 must have a loop too. Since they have the same size, G_1 and G_2 are isomorphic. Similarly, if G_1 has a bigon and no loop, then G_2 must also have a bigon and no loop. Having the same size, G_1 and G_2 are again isomorphic in this case. In summary, if G_1 and G_2 both of rank 2 have the same Ihara zeta function with leading coefficient -3, then G_1 and G_2 are isomorphic.

Next suppose G_1 is a simple graph of rank 2 so that $\zeta_{G_1}(u)^{-1}$ begins with $-4u^{2\|G_1\|}$. Then G is a $G_{m,n,p}$ (with $p > 0$) or a $H_{m,n,l}$ (with $l > 0$). Suppose G_1 is a $G_{m,n,p}$. Further, we may assume $0 < 2p \leq m \leq n$ so that G_1 has girth m.

If $G_2 = H_{m',n',l}$ with $m' \leq n'$ and $l > 0$ has the same Ihara zeta function as G_1, then G_2 has girth m', so $m' = m$. Since they have the same size, we conclude $n' + l = n - p$ or $n = n' + l + p$. Counting spanning trees, we have $mn' = mn - p^2$, whence $p^2 = m(l + p)$. Since $l > 0$ and $m \geq 2p$ this is a contradiction.

Suppose that $G_2 = G_{m',n',p'}$ with $0 \leq 2p' \leq m' \leq n'$ has the same Ihara zeta function as G_1. Since G_1 and G_2 have the same girth, $m' = m$. They must also have the same number of edges, so $m + n - p = m' + n' - p'$, and $n - p = n' - p'$. This means that for such G_2, there is a constant $s = n - p$ for which $n' - p' = s$. So if we vary p', then n' varies equally. If $m = n$, then $p = p'$ and the graphs are the same. Otherwise if $m \neq n$, we can see if the term $-2u^n$ will always be present in the Ihara Zeta Function Inverse by checking if n is strictly less than the powers of the other terms, and strictly greater than m and 0 which is true by $m \neq n$. Since $n > m > 0$, $n > 1$ so $2n > n$. Since $n > m \geq 2p$, we have $m + n - 2p > n$. Since $m > 0$, $m + n > n$. As $m, n > 0$ and $m + n - 2p > n$, we have that $2m + 2n - 2p > n$, $m + 2n - 2p > n$, $2m + n - 2p > n$. We also have $2m + 2n - 4p > 2n$, and since $2n > n$ we know $2m + 2n - 4p > n$. This leaves $2m$ as the only term that could collide with the n term. First assume $n = 2m$. Then the zeta function for $G_{m,n,p}$ is:

$$\begin{aligned}\zeta_{G_{m,n,p}}(u)^{-1} = &- 4u^{6m-2p} + u^{6m-4p} + 2u^{5m-2p} + 2u^{4m-2p} \\ &+ u^{4m} + 2u^{3m} - 2u^{3m-2p} - u^{2m} - 2u^m + 1\end{aligned}$$

We must determine whether the 2m term collides with any other terms. We know $3m - 2p = m + n - 2p > n = 2m$ from earlier, so $m - 2p > 0$. Since $m > 1$, $6m - 2p > 5m - 2p > 4m - 2p > 3m - 2p > 2m$ and likewise $6m > 5m > 4m > 3m > 2m$. Then we also have $2m - 4p > 0$, so $6m - 4p > 4m - 4p > 2m$. This means all exponents will be different than $2m$, so the zeta function will have $-u^{2m} = -u^n$. Because we showed earlier that for $n \neq 2m$ that the zeta function must have term $-2u^n$, this means a graph G_2 has the same zeta function as G_1 only if $n = n'$, and since $n - p = n' - p'$ we know $p = p'$ and $G_2 = G_1$. Second, let $n \neq 2m$. By the previous argument, if $n' = 2m$, then G_2 will have a different zeta function. Then, from our arguments we know the zeta function inverse for G_1 must have term $-2u^n$, and that the zeta function for G_2 must have term $-2u^{n'}$. Therefore it must be the case that $n = n'$, $p = p'$, and $G_1 = G_2$.

There are multigraphs of rank 2 that admit to a similar treatment. We need to lessen the constraints on the definition of a $G_{m,n,p}$ graph to instead have $m, n > p$ (but still $p \geq 0$). The removal of $m, n > 2$ means that when m or n equal one, it will be a loop, and when they equal two a digon. The remaining part of the condition $m, n > p$ ensures that at least one edge will not be combined between the two cycles as opposed to at least two edges.

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